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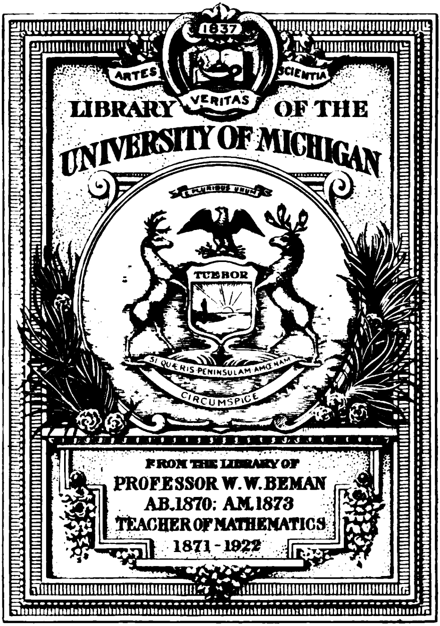

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ALGEBRA

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PART I.

By LANGLEY AND BRADLY

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Mr MURRAY,
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ERRATUM

Page 16, bottom line:—
For "$a+b-c$" *read* "$a-b+c$"

ALGEBRA

ADAPTED TO THE REQUIREMENTS
OF THE FIRST STAGE OF THE
DIRECTORY OF THE BOARD
OF EDUCATION

BY

E.dward M.ann LANGLEY, M.A.

SENIOR MATHEMATICAL MASTER IN BEDFORD
MODERN SCHOOL

AND

S. R. N. BRADLY, M.A.

ASSISTANT MASTER IN BEDFORD MODERN
SCHOOL

LONDON
JOHN MURRAY, ALBEMARLE ST.
1901

Grad. 3.
W. W. Beman
gt.
11-22-1923

6 May 25, EHY

PREFACE

Our object has been to provide at a small price a class-book which shall contain a large number of exercises on the simple rules, with fully-worked examples to show the student how to arrange his own work. Explanations of theory are left in a large measure to the teacher, from whom, in the last resort, all explanations have to come: for experienced teachers are well aware that no matter how full and clear the explanations in a book may be, the ordinary boy will either not read them, or, if he does, will misunderstand them in almost every point. Especially is this likely to be the case in the transition from the Algebra, which is merely generalised Arithmetic, to the Symbolic Algebra, arising from the gradual extensions given to the terms employed. Some of these extensions have already been anticipated in parts of Arithmetic itself, as in the case of Multiplication of Fractions. No pupil who has had explained to him the proper meaning in Arithmetic of $\frac{2}{3} \times \frac{5}{7}$, and why the product is to be taken as $\frac{10}{21}$, will find much difficulty in extending the explanation to the equality $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. While, however, every effort should be made to make the pupil understand the reasons for the processes he employs, we do not believe that the formal proofs of the general theorems involved are fit subjects for writing out until the pupil has made considerable progress in algebraical manipulation. But there should be constant and searching *vivâ voce* questions on all the work done on the board. We hold with Professor Hudson that "Algebra is to be taught as to an intelligent person; . . . people turn out very much as you treat them; . . . a style of teaching that is unintelligent is apt to end in making them so."

Algebraical operations are too apt to be regarded by students as applications of a set of mechanical rules, used without any reference to the realities of the relations which underlie them. It is possible for a teacher to obtain through iteration and practice a certain amount of dexterity in manipulation, which is not without a value of its own as a training in carefulness and neatness. But the study of Algebra should do more for a student than merely train him to work neatly and to be careful in the application of mechanical rules. To quote Professor Hudson again: "The pupil may be instructed merely to perform certain processes and the answer 'comes out.' This is as much teaching Algebra as teaching to grind a barrel organ is teaching music." The teacher should lose no opportunity of bringing home to the minds of his pupils the nature of the operations indicated by the symbols and the truths of the theorems involved in them by illustrations from geometry, from commerce, from the workshop, or from the physical laboratory. He should read what great teachers, such as De Morgan, have written on teaching Algebra, as well as consult the works of later writers, like C. Smith and Chrystal, and provide as much theory for his pupils as he finds they can assimilate; and he must decide for himself as to how far theory is to precede or follow practice. In Part I., therefore, we have supplied *illustrations* rather than *proofs*, and these at somewhat greater length than might seem necessary. We have, however, been unwilling to sacrifice simplicity to brevity. In Part II., dealing mainly with the subjects of the "Second Stage," it is intended to insert a recapitulatory chapter, which will give the *formal proofs* of some principles *illustrated* in Part I. Of the examples a large number are original, but we have not hesitated to draw upon older collections now out of print, and, in the collection of Exercises for Revision, on various school and other examination papers, some of which we have inserted with little or no alteration.

EDWARD M. LANGLEY, M.A.
S. R. N. BRADLY, M.A.

BEDFORD, *July* 1901.

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ALGEBRA

CHAPTER I

USE OF SYMBOLS

1. **Algebra** is a science which treats of numbers. These numbers being represented by symbols, usually letters, it comprises all particular cases in general language, since any desired value may be given to the symbols employed.

2. The letters of the alphabet used as symbols are generally divided into "known" or "given" quantities represented by the earlier letters $a, b, c, \cdot \cdot \cdot,$ while the unknown quantities which have to be found are represented by $x, y, z, \cdot \cdot \cdot,$ etc. Sometimes also the letters of the Greek alphabet, $\alpha, \beta, \gamma,$ are used, as also the forms $a_0, a_1, a_2, \cdot \cdot \cdot,$ and $a', a'', a''',$ to denote different quantities.

Letters are used because they are symbols whose names and forms have already been learned. Any other symbols, such as □ or ⊙, might be used to denote numbers.

3. The connection between different quantities is shown by means of signs, which are certain marks made use of to denote the operations of addition, subtraction, multiplication, division, etc. A symbol or

collection of symbols denoting a number, such as $a+b$, $5\times 3+6$, is spoken of as an *expression.*

4. The sign of Addition (read "plus") is $+$, and shows that the quantity which follows it is to be added to the quantity which precedes it. All quantities to which this sign is prefixed, as well as those which have no sign at all, are called *positive.*

Thus, $12+30$ signifies "12 plus 30," or 12 increased by 30, *i.e.*, 42.

Similarly, $a+b$ signifies "a plus b," or the number represented by "a" increased by the number represented by "b."

5. The sign $-$, which is named minus, is employed to indicate the subtraction of one number from another.

Thus, $54-23$ means "54 minus 23," or 54 diminished by 23, that is to say, 31.

Also, $a-b$ signifies "a minus b," or the number represented by "a" diminished by the number represented by "b."

All quantities preceded by the minus sign are called *negative.*

When an expression denotes a number made up of several other numbers connected by the sign $+$ or $-$, the numbers which thus make it up are called its *terms.*

Thus, $3+5+7$ has 3, 5, and 7 for its terms.

" "$2+9$ times 3" has 2 and 9 times 3 for its terms.

6. When a mathematical operation is to be performed on an expression consisting of more than one term, it is enclosed in brackets, or has a line drawn over it. Thus :—

(i.) $(a-b)-(p+q)$ means that the sum of p and q is to be subtracted from the excess of a over b. If a, b, p, q denote 10, 5, 2, 3 respectively, $(a-b)-(p+q)$ denotes the number obtained by subtracting $2+3$ from the excess of 10 over 5, and is found to be 0.

(ii.) $a+b-\overline{c+d}$ means that $c+d$ is to be treated as a single term, and subtracted from the sum of a and b. If a, b, c, d denote 9, 2, 5, 1 respectively, then $a+b-\overline{c+d}$ denotes the number obtained by subtracting $5+1$ from $9+2$, and is found to be 5.

This line is called a *vinculum*.

7. The sign of Multiplication (read "into," or "multiplied by") is ×.

Thus, 72×26 is read "72 into 26," or "72 multiplied by 26." The result is called, as in arithmetic, the *product* of the multiplicand and multiplier, and these are called the factors of the product.

Similarly, $a \times b$ signifies a multiplied by b, and $a \times b \times c$ signifies that the product of a and b is to be multiplied by c, and so on for any number of factors, the result being called the *continued product* of the factors a, b, c. Each factor of a product is called the coefficient of the product of the other factors.

The process of multiplication is also indicated by placing a dot between the factors. Thus, $a\,.\,b\,.\,c\,.\,d$ means the same as $a \times b \times c \times d$, *i.e.*, the *continued product* of a, b, c, d. The dots are often omitted, and $abcd$ used instead of either of the above.

Similarly, $(a+b)\,(e+f)\,(l+m)$ denotes the *continued product* of $a+b$, $e+f$, and $c+d$.

This notation however can only be used when the numbers are designated by letters, for 54 does not mean 5×4, but fifty-four.

8. The sign ÷ when placed between two numbers signifies that the former is to be divided by the latter.

Thus, $24 \div 8$ means that 24 is to be divided by 8. So also $a \div b$ means that a is to be divided by b.

9. The sign = when placed between two numbers denotes that they are equal to each other.

Thus, $29+17=46$, denotes that the sum of 29 and 17 is equal to 46.

In like manner $a+b=c$ denotes that the sum of a and b is equal to c.

10. A *numerical coefficient* is an arithmetical number which is used to denote that a number designated by some letter or some combination of letters, is added to itself a certain number of times.

Ex.—Instead of writing $a+a+a+a+a+a$, which represents six "*a's*" added together, we write $6a$, and instead of $pq+pq+pq$, we write $3pq$, the numbers 6 and 3, which denote the number of times that the given letter or combination of letters is to be repeated, being called the *numerical coefficients.*

Where no coefficient is expressed, the coefficient 1 is always to be understood. Thus, $1b$ and b represent the same.

11. If each factor of a product is the same, the product is called a *power* of that factor.

Thus, "5×5" is called the "second power" of 5,
„ "$5 \times 5 \times 5$" is called the "third power" of 5,

and so on, the numerical adjective corresponding to the number of times the factor is repeated. The "first power of 5" means the number 5 itself. The words "square" and "cube" are sometimes used as equivalent to "second power" and "third power" respectively. This is owing to some geometrical analogies which the student who has had practice in mensuration will easily perceive.

Similarly, "9×9" is called the "second power" of 9,
„ "$9 \times 9 \times 9$" is called the "third power" of 9,

and so on, the first power of 9 meaning 9 itself.

12. For the sake of saving room and for other reasons,
The second power of 9 is denoted by 9^2,
The third power of 9 is denoted by 9^3,
and so on.

Thus, 9^6 means $9 \times 9 \times 9 \times 9 \times 9 \times 9$, *i.e.*, the continued product of six 9's, and 9^n the continued product of n 9's.

An *index* or *exponent* is the sign employed to indicate that the number designated by a letter is to be multiplied by itself a certain number of times.

Thus, $p \times p \times p \times p$, which represents four "p's" multiplied together, is written p^4, where 4 is called the index of p^4. The index is written a little to the right, above the letter. Where no index is expressed, the index 1 is understood. Thus, q^1 and q signify the same.

13. All numbers written in algebraical language are called *algebraic quantities* or *algebraic expressions*.

Thus, $3a^2 - 6b^2c^3$ is the algebraic expression for the difference between six times the square of b multiplied by the cube of c, and three times the square of a.

EXERCISE I

Express by means of algebraical signs :—

(1) 5 added to 6.
(2) 31 added to 64.
(3) a added to c.
(4) The sum of 11 and 13.
(5) The sum of a, b, and c.
(6) ab added to pq.
(7) a^2 added to b.
(8) $3a$ added to 2.
(9) $2a^2$ added to $3b$.
(10) $21p$ added to $16g$.
(11) 1 added to $30b^2$.
(12) 63 minus 40.
(13) 17 taken from 20.
(14) $2a$ minus 6.
(15) $31a$ taken from $6b$.
(16) $6b$ taken from $31a$.
(17) $2b^3$ minus 5.
(18) 5 minus $2b^3$.
(19) $100a$ taken from 2.
(20) 60 diminished by 5.
(21) 60 diminished by $21a$.
(22) $2b^2$ diminished by 6.
(23) $4ab$ diminished by $2c$.

(24) The remainder when c is taken from 2.

(25) The remainder when 2 is taken from c.

(26) The sum of a and 5 diminished by b.

(27) The sum of 5 and b diminished by a.

(28) The remainder when c is taken from 5 diminished by $16b$.

(29) $21x$ minus the sum of $2a$ and $5b$.

(30) The difference between $3a$ and $5b$.

14. The numerical value of an algebraical expression is the number which results from giving particular values to the letters which compose the expression, and then performing the operations indicated by the signs.

Thus, the numerical value when $a=2$, $b=3$, $c=4$ of

$$3a^2-2ab+3c^2$$

is

$$3\times2\times2 \;-2\times2\times3 \;+3\times4\times4$$
$$=12-12+48 \qquad = 48.$$

15. Terms composed of the same letters with the same indices are called *like terms*.

Thus, $13ac$ and $2ac$ are like terms, as also are $3a^3b$ and $12a^3b$; but $3abc^3$ and $12a^3bc^2$ are not, for though composed of the same letters, the indices are different.

16. The sign $\sqrt{}$ prefixed to an algebraical expression indicates that quantity which, when multiplied by itself, will produce the given expression.

Thus, $\sqrt{25}$ means that number which, when multiplied by itself, produces 25, *i.e.* 5.

$\sqrt{d^2}$ is d, because d when multiplied by itself gives d^2.

So also $\sqrt{(a+b)^2}$ means that quantity which, when multiplied by itself, gives $(a+b)(a+b)$ or $(a+b)^2$ as a product, which is evidently $a+b$.

17. The process indicated above is termed extracting the square root; the result obtained being called the square root of the given expression. Just as a^2, a^4, a^7 are called *powers* of a, so a is called a *root* of any one of these quantities.

Thus, 32 being the "5th power" of 2
 „ 2 is called the "5th root" of 32.
81 being the "4th power" of 3
3 is called the "4th root" of 81.

But the words "square" and "cube" replace the words 2nd and 3rd as applied to roots. Thus, 3 is called the *square* root of 9 (not the *2nd* root), 5 is called the *cube* root of 125 (not its *3rd* root).

The symbols $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[5]{a}$, and so on, denote respectively the cube root, the 4th root, the 5th root, etc., of a.

EXERCISE II

Express in words the following :—

(1) $31+17$.
(2) $105+2$.
(3) $31-17$.
(4) $200-103$.
(5) $a+q$.
(6) $a-q$.
(7) $a+b+c$.
(8) $a-b+c$.
(9) $2a-3b$.
(10) $3p-11q+r$.
(11) $a+5-b$.
(12) $20a+2b-11$.

EXERCISE III

Express by means of algebraical signs :—

(1) The product of 11 and 12.
(2) The product of 5 and 15.
(3) The third part of 18.

(4) 63 divided by 9.

(5) a multiplied by b.

(6) p multiplied by ab.

(7) a^2 multiplied by p^2.

(8) The product of 27 and p.

(9) The product of $2a$ and $4b$.

(10) The continued product of 2, 3, 4.

(11) The continued product of p, q, r.

(12) The square of 11.

(13) The cube of p.

(14) The square of ab.

(15) The fourth power of $(a+b)$.

(16) The eleventh power of ac.

(17) The square root of 64.

(18) The cube root of q.

(19) The square root of a^2b^2.

(20) The fourth root of $a+b$.

EXERCISE IV

When $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$.

Find the numerical value of :—

(1) $a+b-c+d-e+f$.

(2) $3a-2b+3c-4d+5e$.

(3) $20a+20c-20d$.

(4) $a^2+b^2+c^2+d^2$.

(5) $2b^3+3cd-4af$.

(6) $af+bf+cf+df$.

(7) $12abcd-2ef$.

(8) f^2+e^2+2bc.

(9) $d\div 2ab+2e$.

(10) $d\div(2ab+2f)$.

(11) $(a+b)^2+(c+d)^2$. (12) $(b+e)^3-(c+d)^3$.

(13) $\sqrt[3]{2a^2+3b}$. (14) $\sqrt{2d+4b}$.

(15) $\sqrt{(a^2+b^2)e}$. (16) $b\,.\,c\,.\,d-a\,.\,e\,.\,f$.

(17) $(a+2b+3c+4d) \div \sqrt{2+21c-8e}$.

(18) $(e \times a-b \times c+d) \div \sqrt[3]{4e+3c-2}$.

(19) $2a \times 3b \times 4c-12b \div cd$.

(20) $\sqrt{a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd}$.

EXERCISE V

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=6$,

Find the numerical values of:—

(1) $2a+3b-4c+7d$.

(2) $5ab+7bd-8ac+15cd$.

(3) $(12a-b)(25a-3c)(30a-5d)$.

(4) $a^2+b^2-(c+d)^2+3e^2-ef$.

(5) $\sqrt{2(a^2+ab+b^2)+3(2c-d)^2-be}$.

(6) $\sqrt[3]{10a^3+7a^2b-ab^2+3a^2c+bd^2+d^3}$.

18. The correct numerical evaluation of formulæ is a part of Algebra in which pupils who have made considerable progress frequently fail. It is not on this account recommended that teachers should dwell long on it at the outset. On the other hand, we believe it best to push on and reach the more interesting parts of the subject, such as Equations and Equational Problems, as soon as possible. But at frequent intervals the evaluation of formulæ should be insisted on.

Identities such as the following

$$x^2 = x + \ \ 2x\,\frac{x-1}{2}$$

$$x^3 = x + \ \ 6x\,\frac{x-1}{2} + 6x - \frac{x-1}{2}\,\frac{x-2}{3}$$

$$x^4 = x + 14x\,\frac{x-1}{2} + 36x\,\frac{x-1}{2}\,\frac{x-2}{3} + 24x\,\frac{x-1}{2}\,\frac{x-2}{3}\,\frac{x-3}{4}$$

may be found useful, as well as expressions like

$$a^3 + b^3 + c^3 - 3abc,\ \ 2b^2c^2 + 2c^2a^2 + 2a^2b^4 - a^4 - b^4 - c^4,$$
$$(a+b)^3 + 3(a+b)^2c + 3(a+b)^2 + c^3,$$

which vanish when the sum of a, b, and c is zero.

Numerous others will readily occur.

CHAPTER II

ADDITION

19. THE addition of algebraical quantities is performed by combining into one expression all such quantities as are "like" (see § 14), and placing after the result those that are unlike, each preceded by its own sign.

Ex. If we have to add $2ab$ and $3ab$, which are like positive terms, the sum will be $(2+3)$ times ab, *i.e.*, $5ab$. If we have to add $2ab$, $3ab$, and $7ab$, the sum is $(2+3+7)$ times ab, *i.e.*, $12ab$. And generally, to find the sum of any number of like positive algebraical quantities, we take the sum of their coefficients and prefix it to the same quantity.

Similarly, the sum of any number of like negative quantities is the sum of those quantities with the negative sign prefixed.

Ex. The sum of $-2a$, $-5a$, $-12a$ is $-(2+5+12)a$

$$= -19a$$

20. If the sum is required of any number of like quantities, some of which are positive and some negative, we find the sum of all the positive quantities, afterwards that of the negative quantities by adding their coefficients, and then subtract the greater of these sums from the less, prefixing the sign of the greater to the result.

Ex. Find the sum of $2a+3a-4a+5a$.

The sum of the positive quantities is $(2+3+5)a$
$=10a.$

and that of the negative $-4a$,

∴ the sum is $(10-4)a \quad = 6a.$

In the addition of complicated algebraical quantities the process will be greatly assisted if all like terms are written in the same column, each term with its own sign.

Ex. Add together: $5bc + 4ad - 8ac + 5cd + 6ac + 5bc + 2ad - 3mn - 2cd + 3ab - 5ac + 4an + 7an + 2bc - 11ad + 2cd$.

Arranging these in columns, we get

$$\begin{array}{llllllll}
 & 5bc & +4ad & -8ac & +5cd & & & \\
 & +5bc & +2ad & +6ac & & -3mn & & \\
 & & & -5ac & -2cd & & +3ab & +4an \\
 & +2bc & -11ad & & +2cd & & & +7an \\
\hline
\text{Result} & 12bc & -5ad & -7ac & +5cd & -3mn & +3ab & +11an \\
\hline
\end{array}$$

Ex. Simplify: $abc^3 + 6mn^2p^3 + 5abc^2 + 8ab^2c + 11m^2n + abc^3 - 6abc^2 + m^2n^2p^3 + abc^2 - 9ab^2c - 5mn^2p^3 + ab^2c + abc^3 - 2m^2n^2p^3 - 2abc^3 - abc^2$.

Arranging in columns, we have

$$\begin{array}{llllll}
abc^3 & +6mn^2p^3 & +5abc^2 & +8ab^2c & +11m^2n & \\
+\ abc^3 & & -6abc^2 & & & +m^2n^2p^3 \\
 & -5mn^2p^3 & +abc^2 & -9ab^2c & & \\
-2abc^3 & & +abc^2 & +ab^2c & & -2m^2n^2p^3 \\
 & & -abc^2 & & & \\
\hline
 & mn^2p^3 & & & +11m^2n & -m^2n^2p^3 \\
\hline
\end{array}$$

EXERCISE VI

Find the sum of :—

(1) $2+3+11+17+61$

(2) $2+3-11-17+61$

(3) $2-3+11-17+61$

(4) $2a+3a+6a+11a$

(5) $2a-3a+6a+11a$

(6) $12a-3a+6a-11a$

(7) $21x-13x+7x+4x$

(8) $14y^2-11y^2+6y^2-4y^2$

(9) $13ab-41ab+26ab$

(10) $3x^2-5x+7x^2-11x$

(11) $2a^2b^2+4ab^2-3a^2b+6ab^2-4a^2b$

(12) $2x+3y-5z+4x-5y+6z-11x+2y-2y+10x-7y+8z$

(13) $20a+11b-14c+2c-7b+6a-5a+2a-3b-4c$

(14) $6a+5b^2c^3+4a+3d+2b^2c^3$

(15) $7a-5c+3b+2a-3c-7b$

(16) $3b-a-c-115d+6c-5f-d+3a-2b-3c+27e+3e-7f$

(17) $5b-8c+17c-6b-7a+11f-5e+9d+g-3a-5c+2g$

(18) $2ab^2+3ac^2-8cx^2+9b^2x-8hy^2-10ky+5a^2-4ab^2$
$-7bx^2-b^2x-4hy^2-15ky+5ky-hy^2-11x^2+14b^3$
$-22ac^2+10x^2+19ac^2-8b^2x+9x^2+6ky+2hy^2+2ab^2$

(19) $2ab^2+5a^3+14b^3+19ac^2+2ab^2-22ac^2-4ab^2+3ac^2$
$-8cx^2+2a^3-4a^3+21ac^2-6ac^2+2b^3-10ac^2+cx^2$
$+2ac^2-ab^2+7b^3-a^2b+4cx^2$

Where the coefficients are fractions, proceed just as before.

Ex. Add

$$\frac{1}{2}a+\frac{1}{3}b+\frac{1}{4}c,\ \frac{2}{3}a+\frac{1}{6}b-\frac{1}{2}c,\ \frac{1}{4}a-\frac{1}{2}b+\frac{1}{8}c$$

Arranging in columns

$$\begin{array}{rl} & \frac{1}{2}a+\frac{1}{3}b+\frac{1}{4}c \\ & \frac{2}{3}a+\frac{1}{6}b-\frac{1}{2}c \\ & \frac{1}{4}a-\frac{1}{2}b+\frac{1}{8}c \\ \hline \text{Result} & \frac{17}{12}a \qquad\quad -\frac{1}{8}c \\ \hline \end{array}$$

coeff. of a is $\frac{1}{2}+\frac{2}{3}+\frac{1}{4}$

$$=\frac{6+8+3}{12}$$

$$=\frac{17}{12}$$

coeff. of b is $\frac{1}{3}+\frac{1}{6}-\frac{1}{2}$

$$=\frac{2+1-3}{6}$$

$$=0$$

coeff. of c is $\frac{1}{4}-\frac{1}{2}+\frac{1}{8}$

$$=\frac{2-4+1}{8}$$

$$=-\frac{1}{8}$$

N.B.—All rough work should be done at one side of the paper shown up, and *never* on scraps.

EXERCISE VII

Add

(1) $\frac{1}{2}x+\frac{1}{4}y+\frac{1}{8}z,\ \frac{1}{3}x+\frac{1}{4}y-\frac{1}{2}z$

(2) $\frac{2}{3}x-\frac{1}{3}y+\frac{1}{4}z,\ \frac{1}{2}x+\frac{1}{4}y,\ \frac{1}{2}z-\frac{1}{4}x$

(3) $\frac{4}{5}a+\frac{1}{6}b-\frac{1}{2}c,\ \frac{1}{3}b-\frac{1}{2}a+\frac{1}{5}c,\ \frac{1}{4}c-\frac{1}{2}a$

(4) $\frac{2}{3}ab - \frac{1}{6}bc + \frac{1}{2}ac$, $\frac{1}{4}ab + \frac{1}{3}bc - \frac{1}{5}ac$,
$\frac{2}{3}bc - \frac{1}{3}ab$

(5) $2x^2 - \frac{1}{4}x + \frac{5}{6}$, $2 + \frac{1}{2}x - \frac{3}{2}x^2$, $\frac{1}{4}x^2 - \frac{1}{4}x - \frac{7}{6}$

(6) $\frac{8}{15}a^2 + \frac{7}{9}b^2 + \frac{5}{6}c^2$, $\frac{1}{3}a^2 - \frac{2}{3}b^2 + c^2$, $\frac{1}{6}b^2 - \frac{7}{6}c^2 - a^2$

21. We have already shown that the sum of $2a + 3a$ is $(2+3)a$ or $5a$; that is to say, the process of addition has to be applied to the coefficients and the common letter, or letters affixed to the result.

Consequently, if the several quantities to be added, though not fulfilling the definition of "like quantities," yet have one or more letters common to them all, their sum may be found by affixing the common letter or letters to the sum of the rest found as before, and enclosed in a bracket.

Ex. 1. Find the sum of $2ax - 3bx + 4ax$.

The letter x is common to each term, so we find the sum of $2a - 3b + 4a$, *i.e.*, $6a - 3b$. Enclose it in a bracket and add x. Thus, $(6a - 3b)x$.

N.B.—In applying this rule it must be remembered that x^2 and x, though both containing the letter x, have different indices, and must be treated as quite distinct quantities.

Ex. 2. Find the sum of: $7x^3 + 14ax + 13cy + 8x^3 + 5bx - 5ay^2 + 5x^3 + 8ax + 15ay^2 - 11x^2 - 21bx - 10by^2$

Arranging in columns, with the highest power of x first:—

$$\begin{array}{llllll}
 & 7x^3 & & +14ax & +13cy & \\
 & 8x^3 & & +\ 5bx & & -\ 5ay^2 \\
 & 5x^3 & & +\ 8ax & & +\ 15ay^2 \\
 & & -11x^2 & -21bx & & -\ 10by^2 \\
\hline
\text{Result} & 20x^3 & -11x^2 & +(22a-16b)x & +13cy & +(10a-10b)y^2 \\
\hline
\end{array}$$

EXERCISE VIII

Arrange, grouping together the powers of x, y, z, etc., the sum of

(1) $ax^2 + bx + c - cx + bx^2 - a + ax - cx^2 + b$

(2) $8cx^2 + 9b^2x + 2ab^2 - 7bx^2 - b^2x + x^2 - a^3 + a^2x - ab^2$

(3) $2ky^2 + 5ky + 11x - 8b^2x + hy^2 - 3hy + ax + 6$

(4) $ax^3 + bx^2 - cx + d - x^3 - ax^2 + bx - c + dx^3 + x^2 - ax + b$

(5) $2a^2x^2 - 3ax + 1 + x^2 + 3x - 2 - bx^2 + bx + 4$

(6) $ax + by + cz + bx + cy + az + cx + ay + bz - 62abc$

(7) $(a+b)x^2 + (b+c)x + 9 + (b-a)x^2 + (c-b)x + 16$
$+ (a-b)x^2 + (b-c)x - 20$

(8) $ax^3 - bx^2 + cx - d - px^3 + qx^2 - rx + s + bx^3$
$+ cx^2 - qx + r$

(9) $ax^2 + by^2 + cz^2 + bx^2 - ay + az - cx + cy^2 - cz^2 + ax - ay$
$+ bz^2 - 2cx - 2cy - 2cz$

(10) $px^2 + qx + 2y - 3x^2 + 4x - 11y^2 + vy - 8x^2 + 3cy$

SUBTRACTION

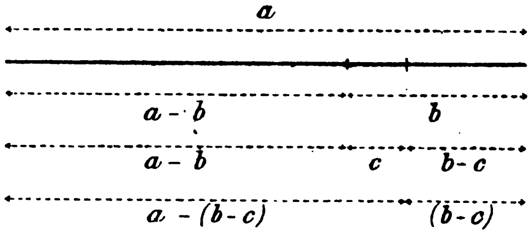
22. To subtract an expression A from a given expression B is to find what expression, called the *remainder*, must be added to A to make B.

Thus, if $A = x - y$, we have to find what quantity added to $x - y$ would make B.

Now on trial it is plain that if to $B - x + y$ we add $x - y$ we get B. Hence, $B - x + y$ is the remainder.

Similarly, if to $B - x + y - z$ we add $x - y + z$, we get B. Hence, $B - x + y - z$ is the remainder when $x - y + z$ is subtracted, and so on for other expressions. The class should be exercised *vivâ voce* with varied expressions, and the equivalence of $a - (b - c)$ and $a + b - c$

should be illustrated on the board by diagrams like those beneath, the students being questioned in detail on the lengths of the lines as they are successively marked off.



The process may also be illustrated concretely.

Thus pupils may be readily brought to see that to subtract 19s. from a given sum of money is the same as to subtract £1 and then add on 1s. to the remainder.

Thus, £5 3s. 6d. − 19s.
= £5 3s. 6d. − (£1 − 1s.)
= £5 3s. 6d. − £1 + 1s.
= £4 3s. 6d. + 1s.
= £4 4s. 6d.

and many similar illustrations with various concrete quantities can be given.

The following exercises, and others like them, should be worked in class.

(i) *Illustrate geometrically*

$$a-(a-b)=b$$
$$a+b-(a-b)=2b$$

(ii) John has £5 and 3s. He pays William £2 all but 4s. How much has he left?

£(5 − 2) and (3 + 4)s., *i.e.* £3 7s.

Ex. 1. From $3ax+4by$ take $2ax+5by$.

Here we have to subtract from $3ax+4by$ a number which exceeds $2ax$ by $5by$. If we first subtract $2ax$ we get $3ax+4by-2ax$ for remainder. But we have to subtract $5by$ more than $2ax$. Hence, to make our result correct, we must subtract $5by$ more, thus obtaining for our final remainder $3ax+4by-2ax-5by$, which is exactly the same as that obtained by changing the signs of the quantity to be subtracted and proceeding as in Addition. That is, "The terms of the expression $2ax+5by$ must be treated as if they were $-2ax-5by$, and then added to $3ax+4by$."

Thus, the remainder is

$$\begin{aligned} &3ax+4by-2ax-5by \\ =\ &(3a-2a)x+(4b-5b)y \\ =\ &ax-by \end{aligned}$$

Ex. 2. From $3ax+4by$ take $2ax-5by$.

If we subtract $2ax$ our result will be

$$3ax-2ax+4by \text{ or } ax+4by$$

but the expression to be subtracted is not $2ax$, but is less than $2ax$ by $5by$. That is to say, we have subtracted $5by$ too much, or in other words, to make the result correct we must add on $5by$, making the result $ax+9by$, which is precisely the effect obtained by changing the signs of the expression to be subtracted, $2ax-5by$, and then proceeding as in Addition.

Thus,

$$\begin{aligned} &3ax+4by-2ax+5by \\ =\ &(3a-2a)x+(4b+5b)y \\ =\ &ax+9by \end{aligned}$$

23. This gives a definite Rule for Subtraction.

Reverse each sign of the expression to be subtracted mentally (*i.e.* in your head), and proceed as in Addition.

N.B.—The signs of the expression to be subtracted should *not* be changed on the paper, but only mentally.

Ex. From $3m - 8x - 7f$ take $3d - 5m - 2x - 6f + 8$.

Arranging in columns as in Addition—

$$\begin{array}{lrrrrr} & 3m & -8x & -7f & & \\ & -5m & -2x & -6f & +3d & +8 \\ \hline \text{Result} & 8m & -6x & -f & -3d & -8 \\ \hline \end{array}$$

EXERCISE IX

(1) From $3a - 8b - 7c$ take $3b - 5a + 2c$

(2) From $24a + 16b + 11c - d$ take $24d + 16c + 11b - a$

(3) From $8a - 5b - 3c - 7d + 5e - 8f$ take $3c - 5b + 2d - 4e - 7f - a$

(4) From $32a + 3b - 5a$ take $27a - 14b$

(5) From $8ac - 5ab - 4bc + 9am$ take $8am + 11ab - 2ac - 7$

(6) From $x^2 + 2x + 1 + a - 2b + y$ take $y^2 + 2y + 1 - 2a + b + x$

(7) From $a + b + a^2 + b^2 + 2ab$ take $2a + 2b + 2a^2 - 2b^2 - ab$

(8) From $24x^2 + 15xy + 16y^2 - 3yz - 3$ take $3xy - 16yz - 5x^2 + 16 + 12y^2$

(9) From $110x - 201y + 3z + a^2 - 2a + b$ take $109b - 200a + 3a^2 - z + 2y + x$.

(10) From $28ax^3 - 16a^2x^2 + 25a^3x - 13a^4$ take $18ax^3 + 20a^2x^2 - 24a^3x - 7a^4$

(11) From $10ax^3 - 36a^2x^2 + 49a^3x - 6a^4$ take $28ax^3 - 16a^2x^2 + 25a^3x - 13a^4$

(12) From $25a^2 - 25x^2 - 23ab - 23cx + 3ac$ take $20a^2 + 20x^2 + 15ab - 15cx + 2ac$

(13) From $6x^3y + 10x^2y^2 + 13xy^3 + 19y^4$ take $5x^3y - 2x^2y^2 + 3xy^3 - 2y^4$

(14) From $x^3y + 12x^2y^2 + 10xy^3 + 21y^4$ take $6x^3y + 10x^2y^2 + 13xy^3 + 19y^4$

(15) From $4a^4 + 6a^3b + 8a^2b^2 + 10ab^3$ take $2a^4 + 3a^3b + 2ab^3 + b^4$

(16) From $2ax + 3by + 4c$ take $2bx + 3ay + 2c - 4$

BRACKETS

24. The use of brackets to enclose two or more terms connected by algebraic signs with the object of denoting that they are to be treated as a single term has already been mentioned.

It occasionally occurs, however, that an expression enclosed within brackets has in its turn to form part of a more extended expression, which itself is enclosed within other brackets. In this case it is usual to employ brackets of different forms. Those in common use are $[\ \ ]$, $\{\ \ \}$, $(\ \ )$.

The expression $x+(y-z)$ signifies that the remainder when z is taken from y is to be added to x, or that y is to be added to x and z subtracted from the result. This is also the meaning attached to $x+y-z$ without any brackets. Consequently it is clear that if an expression enclosed within brackets is preceded by the sign $+$, the brackets may be left out without altering the expression.

A numerical example may make this more clear.

Thus,
$$\begin{aligned} 20+(15-9) &= 20+6 = 26 \\ 20+15-9 &= 35-9 = 26 \end{aligned}$$

which agrees with the results obtained above.

The converse of the above is evidently also true. That is to say, any part of an expression may be enclosed within brackets preceded by the sign $+$ without altering the value of the expression.

Ex. $x+y-z+a-b$ may be written

$$\begin{aligned} & x+(y-z)+a-b \quad \text{or} \quad (x+y)-z+a-b \\ \text{or}\quad & (x+y-z)+a-b \quad \text{or} \quad x+y-z+(a-b) \\ \text{or}\quad & (x+y-z+a)-b \quad \text{or} \quad x+y+(-z+a-b) \end{aligned}$$

etc., etc., without being altered in value at all.

Again, $x-(y+z)$ signifies that the sum of y and z is to be subtracted from x. The same effect is obtained by subtracting y from x, and then z from the remainder.

Thus, $$x-(y+z)=x-y-z$$

Also, $x-(y-z)$ signifies that $(y-z)$ is to be subtracted from x, and by the rules already found for subtraction this is equivalent to changing the signs of both y and z, and adding the results to x.

Thus, $$x-(y-z)=x-y+z$$

Similarly, $20-(15-9)=20-15+9=29-15=14$

25. Hence, we get the following rule:—

When an expression within brackets is preceded by the sign $-$; if the brackets are removed, every sign within the brackets must be changed.

So also,

Any number of terms of an expression may be enclosed in brackets preceded by the sign $-$ without affecting the value of the expression, if the sign of each term within the brackets be changed.

Ex. $x+y-z+a-b$ may be written

$x-(-y+z)+a-b$ or $x-(-y+z-a)-b$

or $x-(-y+z-a+b)$ or $x+y-(z-a+b)$

etc., etc., without in any way altering its value.

26. To remove brackets it is best to begin with the inside pair, and then remove the others in succession, applying the rules given above, which, when put shortly, are:—

(1) If a bracket is preceded by the sign $+$ it may be left out without affecting the signs of the terms inside it.

(2) If a bracket is preceded by the sign $-$ it may be left out, if the signs of all the terms enclosed by it are changed.

Ex. 1. Simplify, by removing brackets and collecting like terms:—

$$x-[y-z-\{x-(y+z-\overline{2x+y-z})\}]$$

Removing the brackets, one pair at a time, we have:—

$$
\begin{aligned}
& x-[y-z-\{x-(y+z-\overline{2x+y-z})\}] \\
= \; & x-[y-z-\{x-(y+z-2x-y+z)\}] \\
= \; & x-[y-z-\{x-y-z+2x+y-z\}] \\
= \; & x-[y-z-x+y+z-2x-y+z] \\
= \; & x-y+z+x-y-z+2x+y-z \\
= \; & 4x-y-z.
\end{aligned}
$$

Ex. 2. Simplify

$$-[12x-\{10y-(12x-y)-(12y-x)\}]$$

Here the two pairs of round brackets ( ) being independent, may be removed together, we have

$$
\begin{aligned}
& -[12x-\{10y-(12x-y)-(12y-x)\}] \\
= \; & -[12x-\{10y-12x+y-12y+x\}] \\
= \; & -[12x-10y+12x-y+12y-x] \\
= \; & -12x+10y-12x+y-12y+x \\
= \; & -23x-y
\end{aligned}
$$

EXERCISE X

Simplify, by removing brackets and collecting like terms:—

(1) $a+b+(c-a)-(b-a)+c$

(2) $c+[b-(a-b+c)]$

(3) $a+b-\{c+[d-(a-b)]\}$

(4) $2a-[3b+(2c-\{3a+b\})]$

(5) $[6a+\{b-c-(2a-b-c)\}]$

(6) $2x-[x+\{3x-(4x-1)\}]$

(7) $-[+\{-(p-q)\}]$

(8) $[x-\{y-x+(z-y)\}]-\{2x-[3x-(2y-z)]\}$

(9) $2ab-[3ab-4bc+\{ac-(bc-2ab)+2ac\}]$

(10) $a+[a-(b+a)-\{a-(b-a)+2b-(a-b)\}]$

(11) $-[-\{-(+[-\{-a\}])\}]$

(12) $10x-[4y-\{4x-(10y-2x)-(3y-x)\}]$

(13) $2a-[a-b-\{c-a+b-(a-b+c-d)\}]$

(14) $4d-\{15c-[2d-(6c-d)-8c]+d\}$

(15) $2b-(a-\overline{b-c})-[2c-\{b-a-(c-b)\}]$

(16) $a-[c-(a+c)-\{a-(b-2a+\overline{b-a})\}]$

(17) $a-[2b+(c-d)+\{b-3d+[c-\{a+c-d-\overline{4a+b}\}]\}]$

(18) $5-[2-\{3-(6-2-\overline{3-1})\}]$

(19) $-[b-\{2c+(d-b)-(d+b)-2a\}+c]$

(20) $2y-[3y-8x-\{2y-(4x-\overline{2y-x})\}+y]$

CHAPTER III

MULTIPLICATION

27. THE multiplication of algebraical quantities is a shorter way of performing the addition of a number of quantities of the same kind, and consequently is carried out according to the common rules of arithmetic.

Ex. 1. "2×5" signifies that the quantity 2 has to be repeated five times, and the five quantities thus obtained are added together.

Ex. 2. "$a\times b$" implies that the positive quantity a has to be repeated b times, and the result is often written ab, which is also a positive quantity.

Ex. 3. $-a\times b$.

If the negative quantity, $-a$, were to be repeated b times, we should get a negative result $-ab$.

Ex. 4. $\overline{a+b}\times c$.

This means that the number a has to be repeated c times, and then the number b has also to be repeated c times, and the two results added together.

Thus, $(a+b)\times c = a\times c + b\times c = ac+bc$

For a geometrical illustration see p. 28.

Ex. 5. $(a+b)\times(c+d)$.

Here we have to take $(a+b)$ c times as above, and then take it d times more, and add the two results together.

$$\begin{aligned}(a+b)\times(c+d) &= (a+b)\times c \;+ (a+b)\times d\\ &= ac+bc \;+ ad+bd\end{aligned}$$

For a geometrical illustration see p. 30.

Ex. 6. $(a-b)\times c.$

If instead of c times the number $a-b$ we take c times the number a, our result will be too great, for each time we take a instead of $\overline{a-b}$, we take b too many. So in taking c times a, we shall have taken c times b too many, and therefore must subtract c times b, or bc, from our result.

$$(a-b)\times c = a\times c \;- b\times c = ac-bc$$

Ex. 7. $(a-b)\times(c-d).$

$\overline{c-d}$ is a number less than c by the number of units in d. Therefore, when we take c times $(a-b)$, we have taken it d times too many, so d times $(a-b)$ has to be subtracted from the result.

$$\begin{aligned}\text{Now,}\quad (a-b)\times c &= ac-bc\\ (a-b)\times d &= ad-bd\\ (a-b)\times c-(a-b)\,d &= ac-bc-ad+bd\\ \text{or}\quad (a-b)\times(c-d) &= ac-bc-ad+bd\end{aligned}$$

For geometrical illustrations see pp. 29, 30.

On examining this result we see that it is obtained by multiplying each term of $a-b$ by each term of $c-d$, putting a + sign where the two terms multiplied together had the same sign, and a − sign where the terms had opposite signs. This is known as *the Rule of Signs in Multiplication*, viz.:—

The product of quantities with like signs is positive.

The product of quantities with unlike signs is negative.

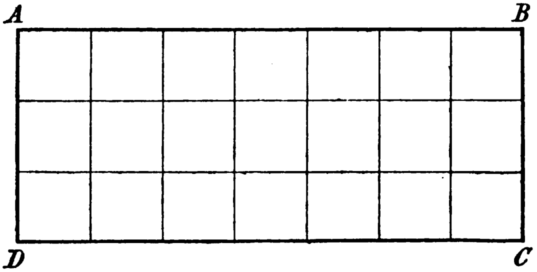
Or, *as an abbreviation for the above* :—

Terms with like signs give +.

Terms with unlike signs give −.

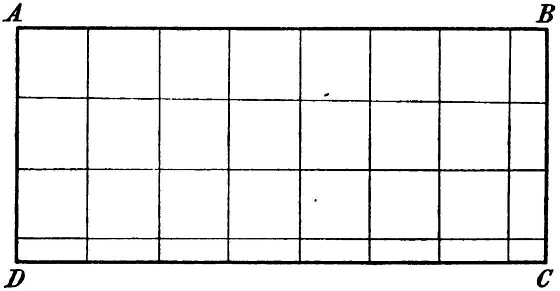
28. We may here advantageously quote De Morgan. "But here the student must avoid the saying that + multiplied by + gives +; − multiplied by + gives −; and so on. He must recollect that the signs + and − are not quantities, but *directions to add and subtract*, and that we might as well say that 'take away' multiplied by 'take away' gives 'add,' as that − multiplied by − gives +. The only way in which the student should accustom himself to state this rule is the following :— *In multiplying two algebraical expressions, multiply each term of the one by each term of the other, and wherever two terms are preceded by the same sign put + before the product of the two; wherever the signs are different put − before the product.*"

29. *Geometrical Illustration of Algebraical Multiplication.*—If the adjacent sides AB, AD of a rectangle ABCD contain respectively 7 units and 3 units of length (inches, centimetres, feet, etc.), it will be found



by ruling parallels to the sides, that the figure can be divided into 3 strips, each containing 7 squares (square inches, square centimetres, square feet, etc.), and hence that the whole rectangle contains just (3 × 7) square units. This is expressed by saying that the area of the figure is (3 × 7) square inches, square centimetres, square feet, etc., as the case may be.

A few more diagrams, in which the numbers of units of AB and AD were varied, would convince the student that the method is correct not only when the sides AB, AD contain an exact number of units of length, but also when they contain fractional numbers like $7\frac{1}{2}$, $3\frac{1}{4}$, as shown in the diagram below.

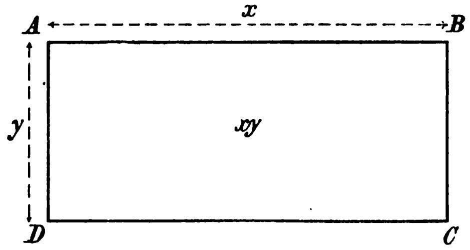


Hence, in all cases, when AB, AD can be arithmetically expressed in terms of the same unit (*e.g.* an inch or a centimetre), the number of square units (square inches or square centimetres) in the area of the rectangle ABCD will be found by multiplying together the numbers which give the lengths of AB, AD. This rule is sometimes, for the sake of brevity, stated as follows:—

To find the area of a rectangle multiply the length by the breadth.

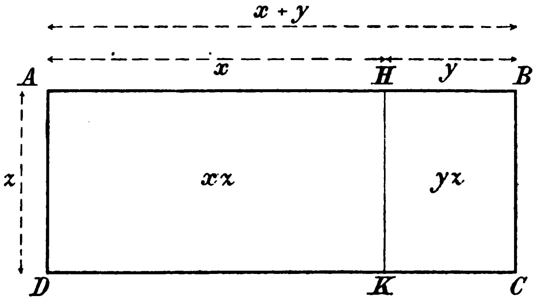
30. Assuming the rule illustrated above, we have a means of representing a product xy *graphically*—*i.e.* by means of a diagram.

Draw AB, AD at right angles to one another, and containing x and y units of length respectively, and complete the rectangle ABCD; then the area of the



rectangle ABCD represents graphically the product xy.

31. *To illustrate geometrically the truth of the theorem* $(x+y)z = xz + yz$,



Draw a straight line AHB, made up of two parts AH, HB, containing x and y units of length respectively, then draw AD at right angles to it, containing

z units of length, and complete the rectangles AHKD, HBCK, forming together the rectangle ABCD; then the area of the whole figure is given by $(x+y)z$, and the areas of the two figures which make it up by xz and yz respectively.

$$\because \quad \text{ABCD} = \text{AHKD} + \text{HBCK}$$
$$\therefore \quad (x+y)z = xz + yz$$

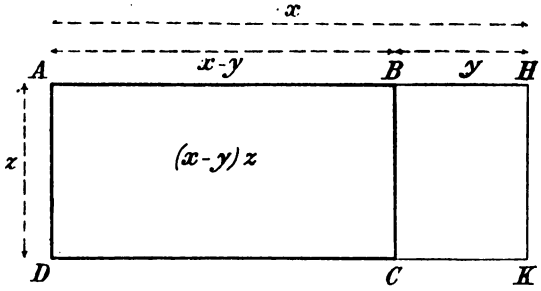
Ex. Illustrate geometrically the theorem

$$p(x+y+z) = px + py + pz$$

32. (i) *To illustrate geometrically the theorem*

$$(x-y)z = xy - yz$$

x being supposed greater than y.

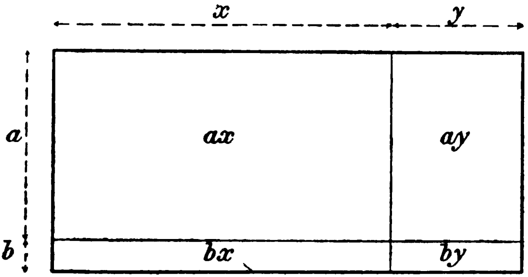


Take AH, HB, containing x and y units of length respectively, but cut off HB from AH; then the remainder, AB, contains $x-y$ units. Draw AD, and complete the rectangles AHKD, HBCK as before.

$$\because \quad \text{ABCD} = \text{ABHK} - \text{BHKC}$$
$$\therefore \quad (x-y)z = xz - yz$$

(ii) *To illustrate geometrically the theorem*

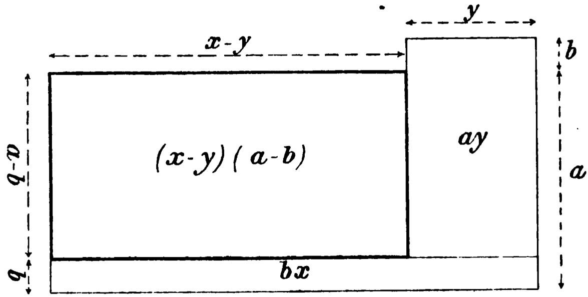
$$(x+y)(a+b)=ax+ay+bx+by$$



The student must state the construction himself, and should be questioned on the properties of the figure.

33. *To illustrate geometrically the theorem*

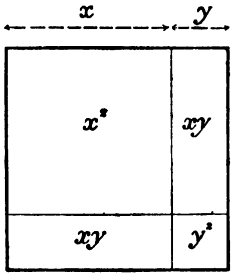
$$(x-y)(a-b)=ax+by-ay-bx$$

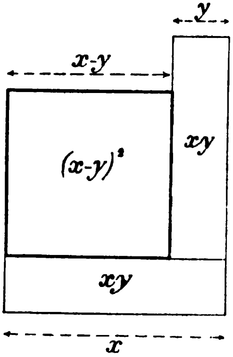


Here the whole figure contains $ax+by$ units, and can also be shown to be made up of figures containing respectively bx, ay, and $(x-y)(a-b)$ unit. Hence, $(x-y)(a-b)=ax+by-bx-ay$.

To illustrate geometrically the theorem

$$(x+y)^2 = x^2 + y^2 + 2xy$$
$$(x-y)^2 = x^2 + y^2 - 2xy$$





It should be noticed that these are special cases of the two previous diagrams, having a and b equal to x and y respectively.

34. We have said that when the same letter occurs several times as a factor in a product, the result may be expressed shortly by writing a number above and to the right of this letter. The number used denotes the number of times the letter occurs as a factor. If, therefore, the same letter occurs in two or more algebraical quantities, which have to be multiplied together, the number of times it occurs in the product will be the sum of the number of times it occurs in each of the quantities, and the index required in the product will be the sum of the indices in the factors.

Thus, $a^2 \times a^3 = a \times a \times a \times a \times a$
$= a^{2+3} = a^5$

Also, $a^3b^2 \times a^4b^7 = a^{3+4}b^{2+7} = a^7b^9$

Or, to express the rule shortly:

"To multiply quantities expressed by the same letter, add the indices of the powers of that letter."

EXERCISE XI

Find the product of:—

(1) 21×17
(2) $3a \times 5bc$
(3) $31a \times 6b$
(4) $2ab \times 3cd$
(5) $5ax \times 17by$
(6) $2a^2r \times 5bc$
(7) $21xy \times 11a^2$
(8) $62ac \times 113by$
(9) $2a \times 5a$
(10) $5b \times 16b$
(11) $20ab \times 2a$
(12) $12a \times 5ax$
(13) $11ab \times 15bc$
(14) $17ac \times 8bc$
(15) $112x^2 \times 7x$
(16) $2a^3 \times 5a^2$
(17) $21b^2 \times 5b^4$
(18) $6b \times 7b^5$
(19) $52b^2c^2 \times 7bc$
(20) $11x^2 \times 9xy$
(21) $5a^2b^2c^2 \times 7abc$
(22) $7a^2c^2d \times 5acd^2$
(23) $14x^2y^3 \times 5axy$
(24) $11x^3y^5 \times 14xy^5z^2$
(25) $-3ab \times 5bc$
(26) $-12xy \times -5xy$
(27) $-21a^3b^3c \times 11a^2b^2c^3$
(28) $17b^2c^2d^3f^2 \times -15a^2bc^3d^2$
(29) $-212abcdx \times -16xyza$

35. To multiply together two algebraical expressions each term of the multiplicand has to be multiplied by each term of the multiplier. All terms to be taken with their own signs, and the rule of signs carefully observed. The products thus obtained to be added together, like terms being collected.

Ex. 1. Multiply $2a^4+5a^3b+6a^2b^2$ by $3a^3-2a^2b-5b^3$

$$\begin{array}{llllll}
2a^4 & +\ 5a^3b & +\ 6a^2b^2 & & & \\
3a^3 & -\ 2a^2b & -\ 5b^3 & & & \\
\hline
6a^7 & +15a^6b & +18a^5b^2 & & & \\
 & -\ 4a^6b & -10a^5b^2 & -12a^4b^3 & & \\
 & & & -10a^4b^3 & -25a^3b^4 & -30a^2b^5 \\
\hline
6a^7 & +11a^6b & +\ 8a^5b^2 & -22a^4b^3 & -25a^3b^4 & -30a^2b^5 \\
\hline
\end{array}$$

Ex. 2.

$$\begin{array}{llllll}
\frac{1}{2}a & -\frac{1}{4}b & +\frac{1}{3}c & & & \\
-\frac{1}{3}a & +\frac{1}{2}b & -\frac{1}{4}c & & & \\
\hline
-\frac{1}{6}a^2 & +\frac{1}{12}ab & -\frac{1}{9}ac & & & \\
 & \frac{1}{4}ab & & -\frac{1}{8}b^2 & +\frac{1}{6}bc & \\
 & & -\frac{1}{8}ac & & +\frac{1}{16}bc & -\frac{1}{12}c^2 \\
\hline
-\frac{1}{6}a^2 & +\frac{1}{3}ab & -\frac{17}{72}ac & -\frac{1}{8}b^2 & +\frac{11}{48}bc & -\frac{1}{12}c^2 \\
\hline
\end{array}$$

$$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$$

$$\frac{1}{3}\times\frac{1}{4}=\frac{1}{12}$$

$$\frac{1}{12}+\frac{1}{4}=\frac{1+3}{12}=\frac{4}{12}=\frac{1}{3}$$

$$-\frac{1}{9}-\frac{1}{8}=\frac{-8-9}{72}=\frac{-17}{72}$$

$$\frac{1}{6}+\frac{1}{16}=\frac{8+3}{48}=\frac{11}{48}$$

N.B.—Rough work should be done on the right hand side of the paper, and not on scraps.

EXERCISE XII

Find the following products:—

(1) $\overline{x+6}\times\overline{x+7}$ (2) $\overline{2x+5}\times\overline{3x+4}$

(3) $\overline{3y+16}\times\overline{5y+11}$ (4) $\overline{a+b}\times\overline{x+y}$

(5) $\overline{2a+3b}\times\overline{5x-4y}$ (6) $\overline{2x-5}\times\overline{3x-4}$

(7) $\overline{3a-11} \times \overline{5a+8}$ (8) $\overline{2x+4y} \times \overline{7x-11y}$

(9) $\overline{a^2-2a+3} \times \overline{3a-6}$ (10) $\overline{x^2+5x-11} \times \overline{7x+8}$

(11) $\overline{3b^2+4b+6} \times \overline{2b^2+b+2}$

(12) $\overline{15x^2+11x-16} \times \overline{3x^2-x}$

(13) $\overline{5x^2-6x+16} \times \overline{11x^2+12x-8}$

(14) $\overline{a^2+ab+b^2} \times \overline{a-b}$

(15) $\overline{a^2-ab+b^2} \times \overline{a+b}$

(16) $\overline{a^2+ab+b^2} \times \overline{a^2-ab+b^2}$

(17) $\overline{a^3+a^2b-ab^2-b^3} \times \overline{a+b}$

(18) $\overline{a^3+a^2b+ab^2+b^3} \times \overline{a-b}$

(19) $\overline{4x-16} \times \overline{4x-16}$

(20) $\overline{4x+16} \times \overline{4x-16}$

(21) $\overline{2x-3y} \times \overline{4x^2+6xy+9y^2}$

(22) $\overline{2x+3y} \times \overline{4x^2-6xy+9y^2}$

(23) $\overline{5x+11} \times \overline{25x^2+55x+121}$

(24) $\overline{5x-11} \times \overline{25x^2-55x+121}$

(25) $\overline{5x-11} \times \overline{25x^2+55x+121}$

(26) $\overline{x^3-y^3} \times \overline{x^3+y^3}$

(27) $(2x+3) \times (3x+4) \times (4x+5)$

(28) $(5x-6) \times (6x+7) \times (7x+8)$

(29) $\overline{x^4+2x^3+3x^2+2x+1} \times \overline{x^2-2x+1}$

36. Just as x^3 denotes the product of three factors, each of which is x (see p. 4), so x^m, *where "m" is some unknown integer or whole number, represents the product of m factors, each of which is x.* These literal indices follow exactly the same rules as the numerical indices.

Thus,
$$\begin{aligned} x^m \times x^2 &= x^{m+2} \\ x^n \times x^n &= x^{n+n} = x^{2n} \end{aligned}$$

37. Two or more terms, when enclosed by a bracket or a vinculum, as $(a-b)$, or $\overline{2x+3y}$, have to be treated as a single term, and if this term were to occur several times as a factor in a product, the same notation as above would be used. Thus, $(p-q)^2$ would mean that the factor $\overline{p-q}$ occurred twice, $(p+q)^3$ that $\overline{p+q}$ occurred three times, etc. Also, by the last section,

$$\begin{aligned} &(p+q)^3 \times (p+q)^2 \\ &\quad = (p+q)^{3+2} = (p+q)^5 \end{aligned}$$

Ex. $(p^2-pq+r)^2$

$$\begin{array}{lllllll}
 & p^2 & -pq & +r & & & \\
 & p^2 & -pq & +r & & & \\
\cline{2-4}
 & p^4 & -p^3q & +p^2r & & & \\
 & & -p^3q & & +p^2q^2 & -pqr & \\
 & & & p^2r & & -pqr & +r^2 \\
\cline{2-7}
\text{Result} & p^4 & -2p^3q & +2p^2r & +p^2q^2 & -2pqr & +r^2 \\
\cline{2-7}
\end{array}$$

EXERCISE XIII

(1) $\overline{a^m+b^n} \times \overline{a^m-b^n}$

(2) $\overline{x^n-y^n} \times \overline{x+y}$

(3) $(x^n+y^n)\times(x^n-y^n)$

(4) $(x^2-y^2)^2$

(5) $(a-b)^2$

(6) $(a+b)^2$

(7) $(a+b)\ (a-b)$

(8) $(2a-3b)^2$

(9) $(3a+2b)^2$

(10) $(3a-2b)^2$

(11) $(x^2+x+1)^2$

(12) $(x^2-x+1)^2$

(13) $(a^2+3a+2)^2$

(14) $(a+b)^3$

(15) $(a-b)^3$

(16) $(2a+3b)^3$

(17) $(3a-2b)^3$

(18) $(a^2-b^2)^2$

(19) $(a^2+b^2)^3$ (20) $x^2+3x+5)^3$

(21) $(4x^2+7xy+6y^2)^2$ (22) $(4x^2+6xy+11y^2)^3$

(23) $(x^2-5ax+6a^2)^3$ (24) $\overline{\tfrac{1}{2}a-\tfrac{1}{3}b}\times\overline{\tfrac{1}{3}a+\tfrac{1}{2}b}$

(25) $\overline{a-\tfrac{1}{2}b+\tfrac{1}{3}c}\times\overline{\tfrac{1}{2}-\tfrac{1}{3}c}$ (26) $\overline{a^2-\tfrac{1}{2}a+\tfrac{1}{4}}\times\overline{a+\tfrac{1}{2}}$

(27) $\overline{\tfrac{1}{4}x^2+\tfrac{1}{6}xy+\tfrac{1}{9}y^2}\times\overline{\tfrac{1}{2}x-\tfrac{1}{3}y}$

(28) $\overline{\tfrac{5}{2}x^2+3ax-\tfrac{7}{3}a^2}\times\overline{2x^2-ax-\tfrac{1}{2}a^2}$

38. The process of multiplying an expression by itself is known as "squaring" it, and the name of *involution* is sometimes given to the operation.

If we consider the result of squaring an expression of the form, $a+b$, we see that the result contains the square of each term of the original expression taken with its own sign, and also a third term, which is twice the product of the two terms which composed the original expression (see p. 31).

$$\begin{array}{ll}
 & a+b \\
 & \underline{a+b} \\
 & a^2+ab \\
 & \underline{\quad\;\; ab+b^2} \\
\text{Result,} & \underline{\underline{a^2+2ab+b^2}}
\end{array}$$

$$\begin{array}{ll}
\text{Again,} & a-b \\
 & \underline{a-b} \\
 & a^2-ab \\
 & \underline{\quad\; -ab+b^2} \\
 & \underline{\underline{a^2-2ab+b^2}}
\end{array}$$

Here the first and last terms are the squares respectively of a and of $-b$, while the middle term is twice the product of a and $-b$ (see p. 31).

From these two examples we get the following rule for squaring an expression consisting of two terms (called a *binomial* expression, or simply a *binomial*):—

Add together *the square of each term* and *twice the product of the two terms*.

In forming the product attention must be paid to the *signs* of the terms.

Thus,

$$\begin{aligned}
(x+2y)^2 &= x^2+2x(2y)+(2y)^2=x^2+4xy+4y^2\\
(a-5)^2 &= (a)^2+(-5)^2+2(a)(-5)\\
&= a^2+25-10a\\
(2a-3)^2 &= (2a)^2+(-3)^2+2(2a)(-3)\\
&= 4a^2+9-12a\\
(4a-3b)^2 &= (4a)^2+(-3b)^2+2(4a)(-3b)\\
&= 16a^2+9b^2-24ab
\end{aligned}$$

N.B. Note that expressions of 3 or more terms can be squared by successive applications of the above formulæ.

e.g.

$$\begin{aligned}
(a+b+c)^2 &= (a+\overline{b+c})^2\\
&= a^2+2a(b+c)+(b+c)^2\\
&= a^2+2ab+2ac+b^2+2bc+c^2\\
(2x+3y-z)^2 &= (2x)^2+4x(3y-z)+(3y-z)^2\\
&= 4a^2+12xy-4az+9y^2-6yz+z^2
\end{aligned}$$

Ex. Draw a diagram illustrating geometrically that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$$

EXERCISE XIV

Write down the square of each of the following :—

(α) $3+5$ (β) $11+12$ (γ) $15-11$ (δ) $17-10$

(1) $(a+c)$
(2) $a-p$
(3) $b+q$
(4) $a+1$
(5) $x+a$
(6) $a-x$
(7) $y-c$
(8) $c+y$
(9) $2x+1$
(10) $2x-3$
(11) $3x+2$
(12) $3x-4$
(13) $2x+y$
(14) $3x-y$
(15) $(2a-3b)$
(16) $3a+2b$
(17) $2b^2+11$
(18) $3x+2y$

(19) $\frac{a}{2} + b$

(20) $\left(\frac{6}{2} - a\right)$

(21) $2a + \frac{6}{2}$

(22) $2b + \frac{a}{3}$

(23) $3c - \frac{4y}{3}$

(24) $\frac{1}{2}x + \frac{1}{4}y$

(25) $\frac{2}{5}x - \frac{1}{2}y$

(26) $5x - \frac{1}{2}a$

(27) $\frac{3}{4}x + 6$

(28) $y^2 - 10x$

(29) $a + b - c$

(30) $3x - 2y + z$

(31) $a + b + c + d$

(32) $a - 2b - 3c$

39. **It is sometimes convenient to arrange an exercise in Multiplication in the following manner :—**

(1)
$$\begin{aligned} & (2x^2 - 3x + 4)\,(x + 3) \\ = {} & 2x^3 - 3x^2 + 4x \\ & \qquad + 6x^2 - 9x + 12 \\ = {} & 2x^3 + 3x^2 - 5x + 12 \end{aligned}$$

(2)
$$\begin{aligned} & (2x^2 - 3x + 4)\,(3x^2 + 5x - 7) \\ = {} & 6x^4 - 9x^3 + 12x^2 \\ & \qquad + 10x^3 - 15x^2 + 20x \\ & \qquad\qquad - 14x^2 + 21x - 28 \\ = {} & 6x^4 + x^3 - 17x^2 + 41x - 28 \end{aligned}$$

(3)
$$\begin{aligned} & (a + bx + cx^2)\,(p + qx + rx^2) \\ = {} & ap + bpx + cpx^2 \\ & \qquad + aqx + bqx^2 + cqx^3 \\ & \qquad\qquad + arx^2 + brx^3 + crx^4 \\ = {} & ap + (bp + aq)x + (cp + bq + ar)x^2 + (cq + br)x^3 + crx^4 \end{aligned}$$

Students should practice this method before trying to factorise expressions of a degree higher than the second. (*N.B.*—The highest power of x in the product obtained in (1) being x^3, the product is said to be of the *third*

degree in x. In (2) and (3) the products are of the *fourth degree in x.*) The process of factorising will frequently consist in reversing the operations performed above.

DETACHED COEFFICIENTS

40. The labour of multiplying together expressions like $5x^2+2x+3$ and $7x^2+4x+1$ may be much shortened by omitting x^2 and x from the work, and only writing down the coefficients of the various terms :—

$$\begin{array}{rrrrr}
5 & +\ 2 & +\ 3 & & \\
7 & +\ 4 & +\ 1 & & \\
\hline
35 & +14 & +21 & & \\
 & +20 & +\ 8 & +12 & \\
 & & +\ 5 & +\ 2 & +3 \\
\hline
35 & +34 & +34 & +14 & +3
\end{array}$$

The powers of x being successive, and the highest in the product being evidently x^4, we have for the product

$$35x^4+34x^3+34x^2+14x+3$$

Similarly, to multiply

$$2x^3+3x^2-4x-5 \text{ by } x^3-6x^2+7x-9$$

we write as follows :—

$$\begin{array}{rrrrrrr}
2 & +\ 3 & -\ 4 & -\ 5 & & & \\
1 & -\ 6 & +\ 7 & -\ 9 & & & \\
\hline
2 & +\ 3 & -\ 4 & -\ 5 & & & \\
 & -12 & -18 & +24 & +30 & & \\
 & & +14 & +21 & -28 & -35 & \\
 & & & -18 & -27 & +36 & +45 \\
\hline
2 & -\ 9 & -\ 8 & +22 & -25 & +\ 1 & +45
\end{array}$$

and obtain for the product

$$2x^6-9x^5-8x^4+22x^3-25x^2+x+45.$$

41. One caution is necessary. If any of the powers of x intermediate between the highest and lowest be missing from multiplicand or multiplier, a zero coefficient must be supplied. Thus, to multiply

$$x^3+3x^2-7 \text{ by } x^3-4x+2$$

we write

$$\begin{array}{rrrrrrr}
1 & +3 & +0 & -7 & & & \\
1 & +0 & -4 & +2 & & & \\
\hline
1 & +3 & +0 & -7 & & & \\
 & & -4 & -12 & +0 & +28 & \\
 & & & +2 & +6 & +0 & -14 \\
\hline
1 & +3 & -4 & -17 & +6 & +28 & -14
\end{array}$$

obtaining for product

$$x^6+3x^5-4x^4-17x^3+6x^2+28x-14$$

Find the coefficient of x^4 in $(1+2x-3x^2+4x^3)^3$

$$\begin{array}{rrrrr}
1 & +2 & -3 & +4 & \\
1 & +2 & -3 & +4 & \\
\hline
1 & +2 & -3 & +4 & \\
 & 2 & +4 & -6 & +8 \\
 & & -3 & -6 & +9 \\
 & & & +4 & +8 \\
\hline
1 & +4 & -2 & -4 & +25 \\
1 & +2 & -3 & +4 & \\
\hline
1 & +4 & -2 & -4 & +25 \\
 & +2 & +8 & -4 & -8 \\
 & & -3 & -12 & +6 \\
 & & & +4 & +16 \\
\hline
 & & & & 39
\end{array}$$

Omitting the coefficients of all powers of x higher than x^4.

The required coefficient is 39.

Find the coefficient of x^8 in $(1+2x-3x^2+4x^3)^3$

Here the term containing x^8 would be the last but one if the full product were written in ascending powers of x. We therefore reverse the order and write thus :—

$$\begin{array}{rr}
4- & 3 \\
4- & 3 \\ \hline
16- & 12 \\
- & 12 \\ \hline
16- & 24 \\
4- & 3 \\ \hline
64- & 96 \\
- & 48 \\ \hline
 & -144
\end{array}$$

Omitting the coefficients of all powers of x lower than x^7.

The required coefficient is -144.

Find the continued product of

$$(x+a),\ (x+b),\ \text{and}\ (x+c)$$

$$\begin{array}{llll}
1+ & a & & \\
1+ & b & & \\ \hline
1+ & a & & \\
 & +\ b & +\ ab & \\ \hline
1+ & (a+b) & +\ ab & \\
1+ & c & & \\ \hline
1+ & (a+b) & +\ ab & \\
 & \quad c & +(ac+bc) & +abc \\ \hline
1+ & (a+b+c) & +(ab+ac+bc) & +abc
\end{array}$$

The required continued product is

$$x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$$

42. Detached coefficients may be used simultaneously

with the arrangement of work described on p. 38. Thus, in Article 39 (2) we might write

$$\begin{array}{crrrrr}
 & \multicolumn{5}{l}{(2-3+4)\,(3+5-7)} \\
= & 6 & -9 & +12 & & \\
 & & +10 & -15 & +20 & \\
 & & & -14 & +21 & -28 \\
= & 6 & +1 & -17 & +41 & -28
\end{array}$$

43. The method of detached coefficients applies to the product of expressions, involving two quantities, x and y, when these are *homogeneous*—*i.e.* when each consists of a number of terms of the same degree. Take for instance

$x^3+3x^2y+5xy^2+by^3$ for multiplicand
and $2x^2+7xy+4y^2$ for multiplier.

Here each term in the multiplicand is of the *third degree*, and each term in the multiplier is of the *second degree*. Hence both multiplier and multiplicand are *homogeneous expressions*, and we may write thus:—

$$\begin{array}{rrrrrr}
1+ & 3+ & 5+ & 6 & & \\
2+ & 7+ & 4 & & & \\
\cline{1-4}
2+ & 6 & +10 & +12 & & \\
+ & 7 & +21 & +35 & +42 & \\
 & + & 4 & +12 & +20 & +24 \\
\hline
2 & +13 & +35 & +59 & +62 & +24
\end{array}$$

and obtain for the product

$$2x^5+13x^4y+35x^3y^2+59x^2y^3+62xy^4+24y^5$$

EXERCISE XV

Examples in Multiplication to be worked by detached coefficients.

(1) Multiply $1+2x+x^2$ by $1+x$

(2) Multiply $1+2x+x^2$ by $1+2x+x^2$

(3) Multiply $1+3x+3x^2+x^3$ by $1+2x+x^2$

(4) Multiply $1+2x+x^2$ by $1-2x+x^2$

(5) Multiply $1+3x+3x^2+x^3$ by $1-3x+3x^2-x^3$

(6) Multiply $1+x+x^2+x^3$ by $1+x$

(7) Multiply $1-x+x^2-x^3$ by $1-x$

(8) Multiply $1+x+x^2+x^3$ by $1-x$

(9) Multiply $1-x+x^2-x^3$ by $1+x$

(10) Multiply a^2-2a+1 by a^2+2a+1

(11) Multiply a^2+a+1 by a^2-a+1

(12) Multiply $x^3+3x^2y+3xy^2+y^3$ by $x^2+2xy+y^2$

(13) Multiply $x^3-3x^2y+3xy^2-y^3$ by $x^2-2xy+y^2$

(14) Multiply $a^2+2ab+2b^2$ by $a^2-2ab+2b^2$

(15) Multiply $x^2+2xy+3y^2$ by $x^2-2xy+3y^2$

(16) Multiply $x^2-4xy+8y^2$ by $x^2+4xy+8y^2$

Find the coefficient of

(17) x^4 in $(1+3x+5x^2+10x^3)^2$

(18) x^5 in $(1-x+x^2-x^3+x^4)^3$

(19) x^4 in $(1-2x+3x^2-4x^4)^3$

(20) x^5 in $(5-4x+7x^2-3x^3)(3+2x-5x^2+6x^3)$

(21) x^6 in $(1+x+x^2+x^3+x^4)^2$

The student being now in a position to solve elementary Simple Equations, it is suggested that he should be allowed to pass on at once to Chapter IX., and try the earlier examples in Exercise XLV. These may be done simultaneously with his work in the easier parts of Division, Factors, and Fractions. As soon as he has mastered the simpler parts of Chapter VIII., he might proceed with Exercises XLVI. and XLVII., without waiting until he has mastered the harder parts of Chapters IV. to VII.

CHAPTER IV

DIVISION

44. **Division** is the inverse operation of multiplication. The words "divisor," "dividend," and "quotient" are used as in arithmetic.

Just as in Arithmetic, when we are told to divide 3996 by 108, we have to find by what number 108 must be multiplied to give 3996 as a product, so in Algebra, if we are told to divide $12a^3b^3c^5$ by $3abc^2$, we have to find by what expression $3abc^2$ must be multiplied to give as a product $12a^3b^3c^5$. To do this we consider separately the numerical and literal factors.

Thus,

3 must be multiplied by 4 to give 12
a ,, ,, a ,, a^2
b ,, ,, b^2 ,, b^3
c^2 ,, ,, c^3 ,, c^5

Hence, the required quotient is $4ab^2c^3$.

45. The rules for finding the quotient may be easily deduced from the fact that the operation of Division is the exact inverse to that of Multiplication.

Ex. In Multiplication, if we have to multiply $a+b+c$ by p, we multiply a, b, c separately by p, obtaining pa, pb, pc, and add up the products thus obtained, giving as the required product, $pa+pb+pc$.

So, also, if we have to divide $qd+qe+qf$ by q, we divide qd, qe, qf separately by q, and add up the quotients obtained — *i.e.* d, e, and f, giving as the required quotient $d+e+f$. Similarly, if we divide $pa-pb+pc$ by p, we obtain a, $-b$, and c, for the separate quotients, and consequently $a-b+c$ for the complete quotient.

The examples worked above might be set down in the same form as in Arithmetic.

(1) $$\begin{array}{r l} q\,) & \underline{qd+qe+qf} \\ & \underline{\underline{\;d+e+f\;}} \end{array}$$

(2) $$\begin{array}{r l} p\,) & \underline{pa-pb+pc} \\ & \underline{\underline{\;a-b+c\;}} \end{array}$$

(3) $$\begin{array}{r l} a^2b\,) & \underline{2a^3b^4-3a^3b} \\ & \underline{\underline{\;2ab^3-3a\;}} \end{array}$$

(4) $$\begin{array}{r l} -3x^2y\,) & \underline{12x^3yz-9x^2y^3z^2} \\ & \underline{\underline{\;-4xz+3y^2z^2\;}} \end{array}$$

EXERCISE XVI

Find the quotients in the following :—

(1) $a^2+2ab+3a^3 \div a$ (2) $b^3+b^4c+b^2c^2 \div b^2$

(3) $b^4c^2+b^2c^4+abc^6+ab^6c \div bc$

(4) $4a^2x^2+5a^3x+6ax^3-4a^4x^4 \div ax$

(5) $abx^2+3a^2b^2x+4a^2b^3x^2+6a^3b^2x^3 \div abx$

(6) $3x^3+6x^2+3ax-15x \div 3x$

(7) $20ax+15ax^2+10ax+5a \div 5a$

(8) $15a^2bc-12acx^2+15ad^2 \div -5ac$

(9) $x^4y^4-7x^3y^3+14x^2y^2-7xy \div xy$

(10) $14ab^3+7a^2b^2-21a^2b^3+35a^3b \div 7ab$

(11) $6a^3b^2-10a^2f+72a^4bx \div 2a^2$

(12) $24a^6+24a^4+6a^2b-30a^2 \div 6a^2$

46. *Long Division.* Both the rule and the reason for it will be better understood from examples worked in Arithmetical and in Algebraical Division.

Arithmetical Example: 856743 ÷ 347

$$
\begin{array}{r}
347)856743(2469\\
\underline{694}\\
1627\\
\underline{1388}\\
2394\\
\underline{2082}\\
3123\\
\underline{\underline{3123}}
\end{array}
$$

The student should notice that in this case

(1) We subtracted 347×2000, *i.e.* 694000 from the given dividend;

(2) We subtracted 347×400, *i.e.* 138800 from 162743 left over;

(3) We subtracted 347×60, *i.e.* 20820 from 23943 still left;

(4) We subtracted 347×9, *i.e.* 3123 from 3123 left over, and had no remainder.

Hence, 856743 must be exactly equal to

$$(2000 + 400 + 60 + 9) \text{ times } 347.$$

i.e. 347 must be multiplied by 2469 to produce 856743 as a product.

Algebraical Example. Divide

$6x^5+20x^4+48x^3+79x^2+78x+63$ by $3x^2+4x+7$

$$\begin{array}{rrrrrrrl}
3x^2+4x+7\,\big) & 6x^5 & +20x^4 & +48x^3 & +79x^2 & +78x & +63\,\big(& 2x^3+4x^2 \\
 & 6x^5 & +\ \ 8x^4 & +14x^3 & & & & +6x+9 \\
\cline{2-4}
 & & +12x^4 & +34x^3 & +79x^2 & & & \\
 & & +12x^4 & +16x^3 & +28x^2 & & & \\
\cline{3-5}
 & & & +18x^3 & +51x^2 & +78x & & \\
 & & & +18x^3 & +24x^2 & +42x & & \\
\cline{4-6}
 & & & & +27x^2 & +36x & +63 & \\
 & & & & +27x^2 & +36x & +63 & \\
\cline{5-7}
\end{array}$$

The student should notice in this case,

(1) We subtracted

$(3x^2+4x+7)\times 2x^3$, *i.e.* $6x^5+8x^4+14x^3$

from the given dividend.

(2) We subtracted

$(3x^2+4x+7)\times 4x^2$, *i.e.* $12x^4+16x^3+28x^2$

from the $12x^4+34x^3+79x^2+78x+63$ left over.

(3) We subtracted

$(3x^2+4x+7)\times 6x$, *i.e.* $18x^3+24x^2+42x$

from the $18x^3+51x^2+78x+63$ still left over.

(4) We subtracted

$(3x^2+4x+7)\times 9$, *i.e.* $27x^2+36x+63$

from $27x^2+36x+63$ left over, and found no remainder.

Hence, $6x^5+20x^4+48x^3+79x^2+78x+63$ must be exactly equal to

$(2x^3+4x^2+6x+9)$ times $(3x^2+4x+7)$

i.e. $(3x^2+4x+7)$ must be multiplied by $2x^3+4x^2+6x+9$ to produce $6x^5+20x^4+48x^3+79x^2+78x+63$

EXERCISE XVII

(1) Divide x^3+6x^2+9x+4 by x^2+2x+1

(2) Divide $a^4+8a^3x+24a^2x^2+32ax^3+16x^4$ by $a+2x$

(3) Divide $x^3+3ax^2+3a^2x+a^3$ by $a^2+2ax+x^2$

(4) Divide $3x^2+8xy+4y^2+10xz+8yz+3z^2$ by $x+2y+3z$

(5) Divide $8a^3+24a^2+18a+4$ by $4a^2+4a+1$

(6) Divide $16b^4+64b^3c+96b^2c^2+64bc^3+16c^4$ by $2b+2c$

(7) Divide $64a^6+32a^4b+8a^2b^2+b^3$ by $4a^2+b$

(8) Divide $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$ by a^2+ab+b^2

(9) Divide $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$ by $x^3+3x^2y+3xy^2+y^3$

(10) Divide $64a^6+192a^5b+240a^4b^2+160a^3b^3+60a^2b^4+12ab^5+b^6$ by $8a^3+12a^2b+6ab^2+b^3$

47. The rules for Multiplication and Subtraction must be strictly adhered to throughout.

In the examples hitherto attempted all the terms have been of the positive sign, and in every case the divisor has been contained an exact number of times in the dividend, without remainder. This is not always the case, as in the following examples:—

Ex. Divide $6x^3-16x^2y+6xy^2+4y^3$ by $3x^2-2xy-y^2$

$$\begin{array}{r|l}
3x^2-2xy-y^2 & 6x^3-16x^2y+6xy^2+4y^3 \;\big(\, 2x-4y \\
 & 6x^3-4x^2y-2xy^2 \\
\cline{2-2}
 & -12x^2y+8xy^2+4y^3 \\
 & -12x^2y+8xy^2+4y^3 \\
\cline{2-2}
\end{array}$$

In this example note,

(1) We subtracted

$$(3x^2 - 2xy - y^2) \times 2x, \text{ or } 6x^3 - 4x^2y - 2xy^2,$$

from the dividend.

(2) We subtracted

$$(3x^2 - 2xy - y^2) \times -4y, \text{ or } -12x^2y + 8xy^2 + 4y^3$$

from the $-12x^2y + 8xy^2 + 4y^3$ left over, and had no remainder.

Hence, $6x^3 - 16x^2y + 6xy^2 + 4y^3$ is exactly equal to $(3x^2 - 2xy - y^2)$ times $(2x - 4y)$,

or that $3x^2 - 2xy - y^2$ must be multiplied by $(2x - 4y)$ to produce $6x^3 - 16x^2y + 6xy^2 + 4y^3$

Ex. Divide $32a^5 - 80a^4b + 80a^3b^2 - 40a^2b^3 + 19ab^4 - b^5$ by $4a^2 - 4ab + b^2$

$$\begin{array}{r|rrrrrr|l}
4a^2 - 4ab & 32a^5 & -80a^4b & +80a^3b^2 & -40a^2b^3 & +19ab^4 & -b^5 & 8a^3 - 12a^2b \\
+b^2 & 32a^5 & -32a^4b & +8a^3b^2 & & & & +6ab^2 - b^3 \\
\cline{2-4}
 & & -48a^4b & +72a^3b^2 & -40a^2b^3 & & & \\
 & & -48a^4b & +48a^3b^2 & -12a^2b^3 & & & \\
\cline{3-6}
 & & & +24a^3b^2 & -28a^2b^3 & +19ab^4 & & \\
 & & & +24a^3b^2 & -24a^2b^3 & +6ab^4 & & \\
\cline{4-7}
 & & & & -4a^2b^3 & +13ab^4 & -b^5 & \\
 & & & & -4a^2b^3 & +4ab^4 & -b^5 & \\
\cline{5-7}
 & & & & \text{Rem.} & +9ab^4 & &
\end{array}$$

That is to say that

$$32a^5 - 80a^4b + 80a^3b^2 - 4a^2b^3 + 19ab^4 - b^5$$

is equal to

$$(8a^3 - 12a^2b + 6ab^2 - b^3) \text{ times } (4a^2 - 4ab + b^2)$$

and $9ab^4$ over.

EXERCISE XVIII

Find the quotients and remainders (if any) of :—

(1) $25a^6 - a^4 - 2a^3 - 8a^2 \div 5a^3 - 4a^2$
(2) $9x^6 - 46x^5 + 95x^2 + 150x \div x^2 - 4x - 5$
(3) $8x^2 - 26xy + 15y^2 \div 4x - 3y$
(4) $48x^3 - 64x^2y + 12xy^2 + 4y^3 \div 12x^2 - 4xy - y^2$
(5) $6b^3 - 32b^2c + 24bc^2 + 32c^3 \div 3b^2 - 4bc - 4c^2$
(6) $a^4 - 9a^2 - 6ab - b^2 \div a^2 + 3a + b$
(7) $c^4 - 6c^3d + 9c^2d^2 - 4d^4 \div c^2 - 3cd + 2d^2$
(8) $81a^4 - 162a^3b + 81a^2b^2 - 4b^4 \div 9a^2 - 9ab + 2b^2$
(9) $12b^4 + 26b^3c - 8b^2c^2 - 10bc^3 - 8c^4 \div 3b^2 + 2bc + c^2$
(10) $1 - 192y^5 + 320y^6 \div 1 - 4y + 4y^2$

EXERCISE XIX

Divide

(1) $a^4 + b^4 - c^4 + 2a^2b^2 + 2c^2 - 1$ by $a^2 + b^2 - c^2 + 1$
(2) $a^5 + a^3b^2 + a^2b^3 + b^5$ by $a^2 - ab + b^2$
(3) $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + 2b^8$ by $a^4 + a^3b + a^2b^2 + ab^3 + b^4$
(4) $p^6 - q^6$ by $p^3 - 2p^2q + 2pq^2 - q^3$
(5) $1 - 10a + 40a^2 - 80a^3 + 80a^4 - 65a^5$ by $1 - 6a + 12a^2$
(6) $1 + x - 17x^8 + 15x^9$ by $1 - 2x + x^2$
(7) $1 - 9x^8 - 8x^9$ by $1 + 2x + x^2$
(8) $8b^3 + 8c^3 + 27a^3 - 36acb$ by $4b^2 + 4c^2 + 9a^2 - 6ac - 6ab - 4bc$
(9) $x^6 + x^4y^2 - 2x^4z^2 - x^2y^4 + x^2z^4 - y^6$ by $x^2 - y^2$
(10) $7x^5 - 5x^3 + 3x^2 - 2x + 1$ by $x^2 - x + 1$

48. The preceding examples have all been arranged in ascending or descending powers of some letter. In cases where this has not been done the student will find it advantageous to arrange both divisor and dividend in either ascending or descending powers of some common letter before proceeding with the process of division.

Ex. 1. Divide $x^4 + 2a^3x - a^2x^2 - a^4$ by $a^2 - ax + x^2$

Arranging both divisor and dividend in descending powers of x we have :—

$$
\begin{array}{rl}
x^2 - ax + a^2\big) & x^4 - a^2x^2 + 2a^3x - a^4\big(x^2 + ax - a^2 \\
 & \underline{x^4 - ax^3 + a^2x^2 \qquad\qquad\quad} \\
 & \qquad + ax^3 - 2a^2x^2 + 2a^3x - a^4 \\
 & \qquad\quad \underline{ax^3 - a^2x^2 \quad + a^3x \qquad} \\
 & \qquad\qquad\quad - a^2x^2 \quad + a^3x \quad - a^4 \\
 & \qquad\qquad\quad \underline{- a^2x^2 \quad + a^3x \quad - a^4} \\
 & \qquad\qquad\quad \overline{\qquad\qquad\qquad\qquad}
\end{array}
$$

Similarly, both divisor and dividend might have been arranged in ascending powers of x.

Ex. 2. Divide $56x^6 + 16x^8 - 32x^7 - 56x^5 + 14x^2 + 49x^4 - 28x^3 + 1 - 4x$ by $1 + 2x^2 - x$.

Arranging both divisor and dividend in ascending powers of x, we have :—

$$
\begin{array}{rl}
 & \qquad\qquad\qquad\qquad \big(\underline{1 - 3x + 9x^2 - 13x^3 + 18x^4 - 12x^5 + 8x^6} \\
1 - x \\ + 2x^2 \Big) & 1 - 4x + 14x^2 - 28x^3 + 49x^4 - 56x^5 + 56x^6 - 32x^7 + 16x^8 \\
 & \underline{1 - x \quad + 2x^2 \qquad\qquad} \\
 & \quad - 3x + 12x^2 - 28x^3 \\
 & \quad \underline{- 3x + \;\; 3x^2 - \;\; 6x^3 \qquad\quad} \\
 & \qquad\qquad 9x^2 - 22x^3 + 49x^4 \\
 & \qquad\qquad \underline{9x^2 - \;\; 9x^3 + 18x^4 \qquad\quad} \\
 & \qquad\qquad\qquad - 13x^3 + 31x^4 - 56x^5 \\
 & \qquad\qquad\qquad \underline{- 13x^3 + 13x^4 - 26x^5 \qquad\quad} \\
 & \qquad\qquad\qquad\qquad\qquad 18x^4 - 30x^5 + 56x^6 \\
 & \qquad\qquad\qquad\qquad\qquad \underline{18x^4 - 18x^5 + 36x^6 \qquad\quad} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad - 12x^5 + 20x^6 - 32x^7 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad \underline{- 12x^5 + 12x^6 - 24x^7 \qquad\quad} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 8x^6 - \;\; 8x^7 + 16x^8 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{8x^6 - \;\; 8x^7 + 16x^8} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad\qquad\qquad\quad}
\end{array}
$$

EXERCISE XX

Divide

(1) $x^9 - 1$ by $x^3 + 1 + x^6$

(2) $216a^3 - 1$ by $1 - 6a$

(3) $a^6 - b^6$ by $b - a$

(4) $21x^3y^2 + 25x^2y^3 + 68xy^4 - 40y^5 - 56x^5 - 18x^4y$ by $5y^2 - 8x^2 - 6xy$

(5) $119c^2 - 96fg + 20f^2 - 113cf - 39d^2 + 72dg + 37df + 408cg - 200cd$ by $-3d + 4f - 17c$

(6) $64y^6 - 16x^3y^3 + x^6$ by $x^2 - 4xy + 4y^2$

(7) $19x^3 - 8x^2 - 15x^4 + x + 1$ by $3x + 1 - 5x^2$

(8) $x^5 + y^5$ by $x^2y^2 + y^4 - xy^3 - x^3y + x^4$

(9) $b^8 + a^4b^4 + a^6b^2 + a^8 + a^2b^6$ by $a^3b + ab^3 + a^4 + b^4 + a^2b^2$

(10) $a^{p+q} + a^qb^p - b^{p+q} - a^pb^q$ by $a^q - b^q$

49. In cases where the coefficients are letters, proceed just as in the preceding examples, grouping together where necessary the coefficients of the different powers of x (*see* Examples below).

Ex. 1. Divide $ax + by + bx + ay$ by $a + b$.

Arranging the quotient in powers of x.

$$\begin{array}{r|l l}
a+b\,) & ax + bx + ay + by & (\,x + y \\
 & \underline{ax + bx } & \\
 & +ay + by & \\
 & \underline{+ay + by} & \\
\end{array}$$

---

50. It is frequently advantageous to work division in the manner shown below:—

To divide $x^3+6x^2+11x+6$ by $x+2$.

$$\begin{aligned} &x^3+6x^2+11x+6\\ =\,&x^3+2x^2\\ &\quad+4x^2+8x\\ &\qquad\qquad+3x+6\\ =\,&\left.\begin{aligned}&x^2(x+2)\\&\quad+4x(x+2)\\&\qquad\qquad+3(x+2)\end{aligned}\right\}A\\ =\,&(x+2)\,(x^2+4x+3)\end{aligned}$$

Quotient $=x^2+4x+3$.

The part marked A may be omitted when the method has become thoroughly understood. The student will notice that we have to write down three expressions, *each of which is divisible by* $x+2$, whose sum shall be x^3+6x^2+11x, and *a term independent of* x. As this term "independent of x" (which has to be put with $3x$) happens to be 6, the given expression is the sum of three expressions, each of which contains $x+2$ as a factor. The given expression must therefore itself contain $x+2$ as a factor. Also, since the three expressions x^3+2x^2, $4x^2+8x$, and $3x+6$ contain $x+2$ respectively, x^2, $2x$, and 3 times the given expression contains $x+2$ altogether x^2+2x+3 times.

51. A teacher with a class will probably find a great difference between the various members of it as to the quickness with which they can understand and apply the above process. A few examples worked on the board with a rapid set of questions to the class as to "what comes next," will do more to secure an intelligent appreciation than any amount of printed explanation, though it is hoped that what is written below may be found useful:—

Starting with x^3, we write $+2x^2$ to complete an expression (x^3+2x^2), which shall be divisible by $x+2$.

We now write down $+4x^2$ to make up the $6x^2$ in the given expression, and put $+8x$ with it to complete an expression $(4x^2+8x)$ which shall be divisible by $x+2$.

We now write down $+3x$ to make up the $11x$ in the given expression, and put $+6$ with it to make an expression $(3x+6)$ which shall be divisible by $x+2$.

To divide $\quad x^3+6x^2+11x+6$ by $x+4$

$$\begin{aligned}
&x^3+6x^2+11x+6\\
&=x^3+4x^2\\
&\qquad +2x^2+8x\\
&\qquad\qquad +3x+12\\
&\qquad\qquad\qquad -\ 6\\
&=(x+4)(x^2+2x+3)-6
\end{aligned}$$

Hence the *quotient* is $\quad x^2+2x+3$
and the *remainder* $\ -6$

To divide $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$ by $x+a$

$$\begin{aligned}
&x^3+(a+b+c)x^2+(bc+ca+ab)x+abc\\
&=x^3+ax^2\\
&\qquad +(b+c)x^2+(b+c)ax\\
&\qquad\qquad +bcx+abc\\
&=(x+a)\{x^2+(b+c)x+bc\}
\end{aligned}$$

Compare page 41.

To divide $x^4+5x^3+9x^2-7x+11$ by x^2-2x+3

$$\begin{aligned}
&x^4+5x^3+9x^2-7x+11\\
&=x^4-2x^3+3x^2\\
&\qquad +7x^3-14x^2+21x\\
&\qquad\qquad +20x^2-40x+60\\
&\qquad\qquad\qquad +12x-49\\
&=(x^2-2x+3)(x^2+7x+20)+12x-49
\end{aligned}$$

Hence the quotient is $\quad x^2+7x+20$
and the remainder is $12x-49$.

52. This method is of great use when we are finding the factors of an expression whose degree is higher than the second. The student should practice it until he is able to use it with rapidity and accuracy. Further examples will be given in the section on Detached Coefficients.

EXERCISE XXI

Divide

(1) $x^3 - 2ax^2 + a^2x - abx - b^2x + a^2b + ab^2$ by $x^2 - ax + bx - ab$

(2) $ax^3 - (a^2 + b)x^2 + b^2$ by $ax - b$

(3) $x^3 - apx^2 + a^2px - a^3$ by $x - a$

(4) $x^5 - px^4 + qx^3 - qx^2 + px - 1$ by $x - 1$

(5) $mpx^3 + (mq - np)x^2 - (mr + nq)x + nr$ by $mx - n$

(6) $x^3 - 2ax^2 + (a^2 - ab - b^2)x + a^2b + ab^2$ by $x - a - b$

(7) $(x^3 - 1)a^3 - (x^3 + x^2 - 2)a^2 + (4x^2 + 3x + 2)a - 3(x + 1)$ by $(x - 1)a^2 - (x - 1)a + 3$

(8) $x(x - 1)a^3 + (x^3 + 2x - 2)a^2 + (3x^2 - x^3)a - x^4$ by $a^2x + 2a - x^2$

(9) $(3c - 6b)a^2 - (c^2 - 4b^2)a + c^3 - 6bc^2 + 12b^2c - 8b^3$ by $c - 2b$

(10) $x^{p+q} + x^qy^q + x^py^p + y^{p+q}$ by $x^q + y^p$

DETACHED COEFFICIENTS

53. In Division as in Multiplication much labour may often be saved by detaching the coefficients from the successive powers of letters in powers of which the divisor and dividend are arranged.

Ex. 1. To divide $x^4 + 3x^3 + 6x^2 + 5x + 3$ by $x^2 + x + 1$

$$
\begin{array}{rrrrrrl}
1+1+1) & 1 & +3 & +6 & +5 & +3 & (1+2+3 \\
 & 1 & +1 & +1 & & & \\
\cline{2-4}
 & & +2 & +5 & +5 & & \\
 & & +2 & +2 & +2 & & \\
\cline{3-5}
 & & & +3 & +3 & +3 & \\
 & & & +3 & +3 & +3 & \\
\cline{4-6}
 & & & & & & \\
\cline{4-6}
\end{array}
$$

Quotient $= x^2 + 2x + 3$.

It is frequently convenient, both in Arithmetical and Algebraical Long Division, to write the quotient *over* the dividend, as in the next example:—

Ex. 2. To divide $2x^3 - 12x^2 + 13x - 15$ by $x - 5$

$$\begin{array}{r|rrrr}
 & 2 & -2 & +3 & \\ \cline{2-4}
1-1 & 2 & -12 & +13 & -15 \\
 & 2 & -10 & & \\ \cline{2-3}
 & & -2 & +13 & \\
 & & -2 & +10 & \\ \cline{3-5}
 & & & +3 & -15 \\
 & & & +3 & -15 \\ \cline{4-5}
 & & & & \\ \cline{4-5}
\end{array}$$

Quotient $2x^2 - 2x + 3$.

Ex. 3. To divide $x^5 - 4x^4 - 101x^3 + 14x^2 + 504x + 576$ by $x^3 - 5x^2 - 6$

$$\begin{array}{r|rrrrrr}
 & & & & 1 & -4 & -96 \\ \cline{5-7}
1+0-5-6 & 1 & -4 & -101 & +14 & +504 & +576 \\
 & 1 & +0 & -5 & -6 & & \\ \cline{2-5}
 & & -4 & -96 & +20 & +504 & \\
 & & -4 & +0 & +20 & +24 & \\ \cline{3-7}
 & & & -96 & +0 & +480 & +576 \\
 & & & -96 & +0 & +480 & +576 \\ \cline{4-7}
 & & & & & & \\ \cline{4-7}
\end{array}$$

Quotient $= x^2 - 4x - 96$.

Ex. 4. To divide $2x^4 - 9xy^3 - 27x^2y^2 + 189xy^3 - 243y^4$ by $(x - 3y)^3$.

We shall divide by $(x - 3y)$ three times in succession, and use the shortened process for division explained on page 53.

$$\begin{array}{rrrrrr} & 2 & -9 & -27 & +189 & -243 \\ = & 2 & -6 & & & \\ & & -3 & +9 & & \\ & & & -36 & +108 & \\ & & & & +81 & -243. \end{array}$$

Again, (the coefficients in first quotient

$$\begin{array}{rrrrr} & 2 & -3 & -36 & +81 \\ = & 2 & -6 & & \\ & & +3 & -9 & \\ & & & -27 & +81. \end{array}$$

Lastly, (the coefficients in the second quotient

$$\begin{array}{rrrr} & 2 & +3 & -27 \\ = & 2 & -6 & \\ & & +9 & -27. \end{array}$$

Hence quotient $=2x+9y$.

Ex. 5. To divide $x^3+(b+c)x^2+(bc+c^2)x+bc^2$ by $x+b$.

$$\begin{array}{rrrrr} & 1 & +(b+c) & +(bc+c^2) & +bc^2 \\ = & 1 & +b & & \\ & & +c & +bc & \\ & & & +c^2 & +bc^2 \end{array}$$

Quotient $=x^2+cx+c^2$.

Many of the examples in Exercise XIV. may be worked with Detached Coefficients.

CHAPTER V

FACTORS

54. WHEN two or more algebraical expressions are multiplied together the result is called the product, and the expressions are said to be factors of that product.

The process of finding the factors of an expression is generally known as *factorizing* or *factoring*, and is of great value in all branches of algebra.

It may be separated into distinct cases.

55. When an expression has a *monomial* factor (*i.e.* a factor consisting of a single term, like bx, $3a^2b$, $7xyz$) this factor can be easily found by inspection, since it must be a factor of each term of the given expression, and the other factor is the quotient, found by dividing the expression by the monomial, as on page 45.

Thus, $abx^3 - bcx^2 + 2abcx = bx(ax^2 - cx + 2ac)$

$$3a^2b + 3ab^2 + 3abc = 3ab(a + b + c)$$

$$14x^2yz + 21xyz^2 = 7xyz(2x + 3z)$$

56. A *binomial* factor (*i.e.* a factor consisting of two terms like $a + b$, $x + 2$, $2x + 3y$) is not so readily seen by inspection.

We proceed to investigate a few rules for the determination of binomial factors of expressions, like $ax^2 + bx + c$ (called expressions *of the second degree* or *quadratic* expressions), or like $ax^3 + bx^2 + cx + d$ (called expressions *of the third degree* or *cubic* expressions).

QUADRATIC EXPRESSIONS

57. In order to understand the method adopted, it is well to build up a quadratic expression by multiplying two binomial factors together, and then examining closely the connection between the factors and the product.

Thus,

$$\begin{array}{l} x+3 \\ \underline{x+4} \\ x^2+3x \\ +4x+4\times 3 \\ \hline x^2+(3+4)x+(4\times 3) \end{array}$$

or $x^2+7x+12$.

That is to say, $x^2+7x+12$ is the product of two factors $x+3$ and $x+4$, and on examining the products we see that the coefficient of x is the sum of 3 and 4, and the term which does not contain any power of x is the product of 3 and 4.

Hence, expressions of the form x^2+bx+c can be split up into factors of the form $x+p$ and $x+q$, if we can find values for p and q such that $p+q=b$ and $pq=c$.

The same investigation applies to expressions of the form cx^2+bx+1, where the factors will be of the form $(px+1)(qx+1)$, when $p+q=b$, $pq=c$.

Thus, $12x^2+7x+1=(4x+1)(3x+1)$.

Ex. 1. $x^2+11x+30$.

If this has factors of the form $x+p$ and $x+q$, the sum of p and q must be 11, and their product 30.

Now the pairs of numbers whose product is 30 are 1 and 30, 2 and 15, 3 and 10, 5 and 6. Therefore, the values of p and q in this case are 5 and 6, and the required factors are $x+5$ and $x+6$,

i.e. $x^2+11x+30=(x+5)(x+6)$

Neither $1+30$, $2+15$, nor $3+10$ makes 11, but $5+6=11$.

Similarly, $30x^2+11x+1=(6x+1)(5x+1)$

Ex. 2. $x^2-13x+30$.

Hence, assuming the factors as before to be $x+p$ and $x+q$, we have $p+q=-13$, and $pq=30$.

Now, since the sum of p and q is -13, it is clear that one at least of the numbers p and q is negative, and since their product is $+30$, and by the rule of signs like signs give plus, both must in this case be negative, looking for a pair of factors, as in the previous example, whose sum is -13, we get -10 and -3 for p and q, and the factors are $x-10$ and $x-3$.

Similarly, $30x^2-11x+1=(6x-1)(5x-1)$.

It is clear that if the factors of x^2+ax+b are $x+p$ and $x+q$, where $p+q=a$ and $pq=b$, then the factors of $x^2+axy+by^2$ will be $x+py$ and $x+qy$, for their product is $x^2+(p+q)xy+pqy^2$, and $p+q$, we know, $=a$, and $pq=b$.

Therefore, $(x+py)(x+qy)=x^2+axy+by^2$.

With the same conditions, $p+q=a$, $pq=b$,

$$(px+y)(qx+y)=bx^2+axy+y^2$$

Ex. $x^2+17xy+30y^2$.

Let the factors be $x+py$ and $x+qy$.

Then, p and $q=17$, $pq=30$, and the pairs of numbers whose product is 30 are

1 and 30, 2 and 15, 3 and 10, and 5 and 6,

and the second is the only pair whose sum is 17.

Therefore, factors are :—

$$(x+2y)(x+15y)$$

EXERCISE XXII

Factorize :—

(1) x^2+3x+2

(2) x^2+4x+4

(3) a^2+5a+6

(4) a^2+6a+5

(5) $p^2+11p+30$
(6) $x^2+13x+36$
(7) $y^2+17y+66$
(8) $b^2+15b+54$
(9) $a^2+21a+110$
(10) $x^2+14x+49$
(11) $x^2+14x+24$
(12) $c^2+25c+126$
(13) $y^2+24y+63$
(14) $x^2+16x+64$
(15) $x^2+20x+64$
(16) $b^2+34bx+33x^2$
(17) $a^2+29ax+210x^2$
(18) $x^2+8xy+7y^2$
(19) $q^2+104qx+400x^2$
(20) $t^2+62ty+960y^2$
(21) $b^2+27bc+152c^2$
(22) $y^2+28yz+160z^2$
(23) $y^2+28yz+196z^2$
(24) $x^2+53xy+196y^2$
(25) $x^2+26xy+69y^2$
(26) $x^2+30xy+225y^2$
(27) $y^2+32yz+256z^2$
(28) $a^2+30ab+224b^2$
(29) $q^2+27qr+92r^2$
(30) $x^2+40xz+256z^2$
(31) $2x^2+3x+1$
(32) $4x^2+4x+1$
(33) $6a^2+5a+1$
(34) $5a^2+6a+1$
(35) $33b^2+34bx+x^2$
(36) $69x^2+26xy+y^2$

EXERCISE XXIII

Factorize :—

(1) a^2-2a+1
(2) b^2-7b+6
(3) a^2-8a+7
(4) $x^2-15x+36$
(5) $x^2-14x+49$
(6) $x^2-12x+36$
(7) $y^2-13y+12$
(8) $y^2-11y+28$
(9) $x^2-9x+20$
(10) $c^2-16c+48$
(11) $a^2-16a+28$
(12) $b^2-17b+30$
(13) $x^2-19x+90$
(14) $y^2-17y+66$
(15) y^2-3y+2
(16) $y^2-11yz+10z^2$
(17) $x^2-14xy+45y^2$
(18) $x^2-18xz+81z^2$
(19) $t^2-22tr+121r^2$
(20) $p^2-43pq+120q^2$

(21) $x^2 - 11xy + 30y^2$

(22) $x^2 - 22xy + 40y^2$

(23) $x^2 - 13xt + 40t^2$

(24) $x^2 - 20xu + 100u^2$

(25) $a^2 - 20ab + 36b^2$

(26) $a^2 - 9ab + 18b^2$

(27) $b^2 - 24bc + 80c^2$

(28) $x^2 - 15xy + 14y^2$

(29) $x^2 - 26xy + 169y^2$

(30) $x^2 - 30xy + 201y^2$

(31) $6b^2 - 7b + 1$

(32) $7a^2 - 8a + 1$

(33) $36x^2 - 15x + 1$

(34) $49x^2 - 14xy + y^2$

(35) $36a^2 - 12ab + b^2$

(36) $12y^2 - 13yz + z^2$

Ex. 3. $x^2 + 3x - 40$.

Now, the pairs of numbers whose product is 40 are
2 and 20, 4 and 10, 5 and 8.

Of these the last pair 5 and 8 differ by 3, and since the difference is $+3$, the larger of the two numbers, *i.e.* 8, must have $+$, and the smaller 5 must have $-$ prefixed.

Therefore, factors required are:—

$$(x+8)(x-5).$$

Ex. 4. $x^2 - 6x - 40$.

The pairs of numbers whose product is 40 are
2 and 20, 4 and 10, 5 and 8,
but the sum is to be -6, therefore the numbers required are 4 and -10, and factors are $(x+4)(x-10)$.

Similarly, $-40x^2 - 6x + 1 = (4x+1)(-10x+1)$
and hence, $40x^2 + 6x - 1 = (4x+1)(10x-1)$.

EXERCISE XXIV

Factorize:—

(1) $x^2 - x - 6$

(2) $x^2 + x - 12$

(3) $x^2 + 2x - 15$

(4) $x^2 - 2x - 35$

(5) $x^2 - x - 30$

(6) $x^2 + 3x - 4$

(7) $x^2 - 6x - 7$

(8) $y^2 - 2y - 63$

(9) $a^2+7a-18$

(10) $b^2-5b-14$

(11) b^2+b-30

(12) $b^2+2b-24$

(13) $y^2-4y-45$

(14) $y^2+3y-28$

(15) $y^2-5y-84$

(16) $x^2+2xy-99y^2$

(17) $x^2+5xz-24z^2$

(18) $x^2-6xt-72t^2$

(19) $a^2-5az-150z^2$

(20) $a^2-7ab-30b^2$

(21) $a^2-ac-132c^2$

(22) $b^2+by-156y^2$

(23) $b^2-2bc-143c^2$

(24) $b^2+4bd-21d^2$

(25) $x^2-6xy-40y^2$

(26) $x^2-5xy-24y^2$

(27) $x^2-7xz-78z^2$

(28) $y^2-3yz-154z^2$

(29) $y^2-11yz-42z^2$

(30) $y^2-15yz-100z^2$

(31) $6x^2+x-1$

(32) $12x^2-x-1$

(33) $15x^2-2x-1$

(34) $35x^2-2xy-y^2$

(35) $30x^2-xy-y^2$

(36) $156y^2-by-b^2$

EXERCISE XXV

Factorize :—

(1) $x^2-7x-18$

(2) $x^4-2x^2y^2+y^4$

(3) $x^2-xy-2y^2$

(4) $x^2+(a+b)x+ab$

(5) $x^2-ax+bx-ab$

(6) $x^2-xy-12y^2$

(7) $y^2-3yz-28z^2$

(8) x^4-6x^2-16

(9) $a^2-16a+63$

(10) $y^2+14y+24$

(11) $p^2-16p-36$

(12) $p^2+(c+d)p+cd$

(13) $q^2+cq-q-c$

(14) $q^2+q-cq-c$

(15) $b^2+cb+db+cd$

(16) $b^2+cb-db-cd$

(17) $a^2+29a+180$

(18) $a^2-29a-30$

(19) a^4-16a^2+64

(20) a^4-24a^2-25

(21) $x^2-17x+72$

(22) $x^2-17x-60$

(23) $x^2-bx-cx+bc$

(24) $x^2y^2-15xy-216$

58. A commonly occurring form of expression to be resolved into factors is that which consists of the difference of two squares, such as $x^2 - d^2$.

From what has been previously done, we see that the factors must be of the form $(x+p)$ and $(x+q)$, where $p+q=0$ and $pq=-b^2$.

This is satisfied if $p=d$, $q=-d$, and the factors are

$$(x+d)(x-d).$$

Hence, we see that when the expression *to be factorized* consists of the difference of the squares of two quantities, the factors are respectively the sum and difference of these quantities. This is frequently of use in finding the difference of the squares of two numerical quantities.

Ex. 1. $(329)^2-(318)^2$

$$\begin{aligned} &=(329-318)(329+318)\\ &=(11)(647)\\ &=7117\end{aligned}$$

Ex. 2. Factorize $225x^2-81y^2$

$$\begin{aligned} &225x^2-81y^2\\ &=(15x)^2-(9y)^2\\ &=(15x-9y)(15x+9y)\end{aligned}$$

EXERCISE XXVI

Resolve into factors, and find the value of:—

(1) $(118)^2-(96)^2$

(2) $(314)^2-(197)^2$

(3) $(326)^2-(324)^2$

(4) $(102)^2-(90)^2$

(5) $(204)^2-(201)^2$

(6) $(394)^2-(116)^2$

(7) $(24)^2-(11)^2$

(8) $(1003)^2-(9)^2$

(9) $(1901)^2-(1887)^2$

EXERCISE XXVII

Resolve into factors:—

(1) $x^2 - 16$

(2) $y^2 - 9$

(3) $y^4 - 25$

(4) $x^6 - 121$

(5) $25 - a^2$

(6) $100 - b^2$

(7) $4y^2 - 81$

(8) $49 - 9b^2$

(9) $a^4 - b^2$

(10) $b^4 - a^2$

(11) $x^2 - 16y^2$

(12) $16a^2 - 9b^2$

(13) $36x^2 - 49y^2$

(14) $81y^4 - 100$

(15) $196 - 144x^2$

(16) $9y^2 - 169$

(17) $p^2q^2 - 16$

(18) $1 - 4p^2$

(19) $p^4 - 16$

(20) $49a^2b^2 - c^2$

(21) $9 - 9a^2$

(22) $a^2b^2 - 1$

(23) $144x^8 - 81y^{16}$

(24) $a^4b^4c^4 - p^2$

(25) $121a^8b^2c^{12} - 25d^2$

(26) $c^{16} - a^8$

(27) $a^{12} - 9b^8$

(28) $y^{14} - 1$

(29) $4a^2b^2 - 144c^2$

(30) $81c^6 - 4$

59. Expressions which do not appear at first sight to belong to either of the four classes,

$$x^2 + bx + c, \quad cx^2 + bx + 1, \quad x^2 + bxy + cy^2, \quad cx^2 + bxy + y^2,$$

may sometimes be reduced to one of them without much trouble. Thus:—

$$\begin{aligned} 25x^2 + 35x + 12 &= (5x)^2 + 7(5x) + 12 \\ &= (5x + 3)\,(5x + 4) \\ 7x^2 + 24xy + 9y^2 &= 7x^2 + 8x(3y) + (3y)^2 \\ &= (7x + 3y)\,(x + 3y) \\ 16x^2 - 8x - 3 &= (4x)^2 - 2(4x) - 3 \\ &= (4x + 1)\,(4x - 3) \\ 3x^2 + 14xy - 49y^2 &= 3x^2 + 2x(7y) - (7y)^2 \\ &= (3x - 7y)\,(x + 7y) \end{aligned}$$

EXERCISE XXVIII

Find the factors of:—

(1) $25x^2+15x+2$ (2) $2x^2+21x+49$

(3) $9a^2+39a+22$ (4) $49y^2-91y+12$

(5) $7a^2-16a+4$ (6) $36a^2-60ab+25b^2$

(7) $25x^2-5x-6$ (8) $49x^2+7x-12$

(9) $121y^2-33y-28$ (10) $a^2+2ab+b^2+5a+5b+6$

(11) $a^2-2ab+b^2+7a-7b+12$

(12) $x^2+2xy+y^2-x-y-2$

60. To factorize expressions of the form ax^2+bx+c when neither a nor c is unity.

As was shown on page 65, examples of this class may sometimes be reduced to one of the simpler cases investigated on pp. 59-62. Sometimes, too, they fall at once under one of the three theorems:—

$$\left.\begin{array}{ll}(1) & A^2+2AB+B^2=(A+B)^2 \\ (2) & A^2-2AB^2+B^2=(A-B)^2\end{array}\right\} \text{ see pp. 31, 36.}$$

$$(3) \quad A^2-B^2=(A+B)(A-B) \quad \text{see p. 64.}$$

For instance,

$$\begin{aligned}4x^2+12xy+9y^2 &= (2x)^2+2(2x)(3y)+(3y)^2 \\ &= (2x+3y)^2 \\ 25x^2-20xy+4y^2 &= (5x)^2-2(5x)(2y)+(2y)^2 \\ &= (5x-2y)^2 \\ 16x^2-49y^2 &= (4x)^2-(7y)^2 \\ &= (4x+7y)(4x-7y)\end{aligned}$$

Before proceeding to factorize expressions which cannot be thus reduced, it will be best to examine how such forms may arise by the multiplication of two factors.

$$\begin{array}{r}
3x + 4 \\
\underline{2x + 5} \\
6x^2 + 8x \qquad \\
\underline{+15x + 20} \\
6x^2 + 23x + 20
\end{array}$$

If we multiply $3x+4$ by $2x+5$ we obtain as the product

$$6x^2+23x+20$$

when $6 = 2 \times 3$

$20 = 4 \times 5$

and $23 = (4 \times 2) + (3 \times 5)$

and, in general, if we multiply $px+q$ by $lx+m$ and arrange the product in the form ax^2+bx+c, we have

$$lp=a, \quad mq=c, \quad lq+mp=b.$$

Conversely, if in any case where the values of a, b, and c are given numerically, we can guess numbers l, m, p, q, which satisfy these equations, we are able to write down the factors of

$$ax^2+bx+c$$

For instance, if we are asked to factorize

$$12x^2+23x+10,$$

we have to choose for l and p two numbers whose product is 12, and for m and q two numbers whose product is 10, such that $lq+mp=23$.

On trial we find that $l = 3, \quad p = 4$

$m = 2, \quad q = 5$

satisfy all three conditions. (See Article 65 for the treatment of cases in which l, m, p, q are not readily guessed.)

Hence, $12x^2+23x+10=(4x+5)(3x+2)$

Similarly,
$$\begin{aligned}
12x^2+23xy+10y^2 &= (4x+5y)(3x+2y) \\
10x^2-41x+21 &= (5x-3)(2x-7) \\
10x^2+33xy-7y^2 &= (2x+7y)(5x-y)
\end{aligned}$$

One special case is worthy of notice,

$$\begin{aligned}
(ax+by)(bx+ay) &= abx^2+(a^2+b^2)xy+aby^2 \\
&= ab(x^2+y^2)+(a^2+b^2)xy
\end{aligned}$$

e.g. if we are given $2(x^2+y^2)+5xy$ to factorize, we have to guess two numbers whose product is 2 and the sum of whose squares is 5. It is easy to see that 1 and 2 satisfy these conditions.

Hence, $2x^2 + 5xy + 2y^2 = (2x + y)(x + 2y)$

So, $6x^2 - 13xy + 6y^2 = (3x - 2y)(2x - 3y)$

EXERCISE XXIX

Factorize :—

(1) $15x^2 + 13x + 2$
(2) $2x^2 + 11x + 5$
(3) $21x^2 + 41x + 10$
(4) $12x^2 - 23x + 10$
(5) $6x^2 + 13xy + 6y^2$
(6) $10x^2 - 33xy - 7y^2$
(7) $2x^2 - 5x + 2$
(8) $15x^2 - 13xy + 2y^2$
(9) $2x^2 - 13xy + 15y^2$
(10) $2x^2 - 9xy + 9y^2$
(11) $2(a+b)^2 + 9(a+b)(c+d) + 9(c+d)^2$
(12) $15 + 13(a+b) + 2(a+b)^2$
(13) $25x^2 + 20xy + 4y^2$
(14) $4x^2 - 12xy + 9y^2$
(15) $4x^2 - 20xy + 25y^2$
(16) $49x^2 - 16y^2$
(17) $2(a+b)^2 + 5(a+b)(c+d) + 2(c+d)^2$
(18) $6a^2 - 13a(c+d) + 6(c+d)^2$
(19) $4(a+b)^2 + 12(a+b)(c+d) + 9(c+d)^2$
(20) $25(a+b)^2 - 20(a+b)(c+d) + 4(c+d)^2$

FACTORIZATION OF $ax^3 + bx^2 + cx + d$.

61. There are a few standard cases which should be verified by multiplication and committed to memory.

(1) $x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3$
(2) $x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3$
(3) $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$
(4) $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
(5)
$x^3 + a^3 + b^3 - 3abx = (x+a+b)(x^2 + a^2 + b^2 - ax - bx - ab)$

Examples—

$$x^3+6x^2+12x+8=x^3+3\,.\,2x^2+3\,.\,2^2x+2^3$$
$$=(x+2)^3$$

$$27y^3-27y^2z+9yz^2-z^3=(3y)^3-3z(3y)^2+3z^23y-2^3$$
$$=(3y-z)^3$$

$$x^3+8=x^3+2^3$$
$$=(x+2)\,(x^2-2x+2^2)$$

$$125z^3-1=(5z)^3-1^3$$
$$=(5z-1)\,\{(5z)^2+5z+1\}$$
$$=(5z-1)\,(25z^2+5z+1)$$

$$8a^3+b^3-c^3+6abc=(2a)^3+b^3+(-c)^3-3\,.\,2ab(-c)$$
$$=(2a+b-c)\,(4a^2+b^2+c^2-2ab+2ac+bc)$$

62. There are other cases in which the cubic is seen to be made up of two groups, which each have the same factor.

For instance, in $2x^3+3x^2+4x+6$, each of the two groups, $2x^3+3x^2$, $4x+6$, which make it up, is seen to have factor $2x+3$.

Hence, $2x^3+3x^2+4x+6=x^2(2x+3)+2\,(2x+3)$
$$=(2x+3)\,(x^2+2)$$

Similarly,

$$3x^3-4x^2y-9xy^2+12y^3=x^2(3x-4y)-3y^2(3x-4y)$$
$$=(3x-4y)\,(x^2-3y^2).$$

Again, in $x^3+5x^2+10x+8$, both x^3+8 and $5x^2+10x$ are seen to have $x+2$ as a factor.

Hence,

$$x^3+5x^2+10x+8=x^3+8+5x^2+10x$$
$$=(x+2)\,(x^2-2x+4)+5x\,(x+2)$$
$$=(x+2)\,(x^2-2x+4+5x)$$
$$=(x+2)\,(x^2+3x+4).$$

Similarly,

$$27y^3-21y^2+28y-64=(3y)^3-4^3-(21y^2-28y)$$
$$=(3y-4)\,(9y^2+12y+16)-7y(3y-4)$$
$$=(3y-4)\,(9y^2+12y+16-7y)$$
$$=(3y-4)\,(9y^2+5y+16).$$

63. If the cubic given to be factorized cannot be reduced to one of the "standard forms," and cannot be seen to be made up of two groups with a common factor, an attempt should be made to arrange it in three groups, which each contain the same factor, by the process recommended for Division on page 53.

Example.—If we are asked to factorize

$$x^3+10x^2+29x+24,$$

we select some binomial factor of the form $x+p$ as a trial divisor. If it is to be a factor of the given expression, p must be some factor of 24. We will try $x+1$, $x+2$, $x+3$ in succession by the method of pp. 53, 54.

$$\begin{aligned}&x^3+10x^2+29x+24\\ =&x^3+x^2\\ &\quad+9x^2+9x\\ &\qquad\quad+20x+20\\ &\qquad\qquad\quad+4.\end{aligned}$$

Hence, $x+1$ leaves a remainder 4.

$$\begin{aligned}&x^3+10x^2+29x+24\\ =&x^3+2x^2\\ &\quad+8x^2+16x\\ &\qquad\quad+13x+26\\ &\qquad\qquad\quad-2.\end{aligned}$$

Hence, $x+2$ leaves a remainder -2.

$$\begin{aligned}&x^3+10x^2+29x+24\\ =&x^3+3x^2\\ &\quad+7x^2+21x\\ &\qquad\quad+8x+24\\ =&(x+3)(x^2+7x+8).\end{aligned}$$

As we cannot find two numbers whose sum is 7 and whose product is 8, we cannot factorize the quadratic x^2+7x+8.

Similarly,

$$\begin{aligned}&x^3+6x^2+11x+6\\ &=x^3+x^2\\ &\qquad+5x^2+5x\\ &\qquad\qquad+6x+6\\ &=(x+1)(x^2+5x+6)\\ &=(x+1)(x+2)(x+3).\end{aligned}$$

$$\begin{aligned}&6x^3+25x^2+32x+12\\ &=6x^3+9x^2\\ &\qquad+16x^2+24x\\ &\qquad\qquad+8x+12\\ &=(2x+3)(3x^2+8x+4)\\ &=(2x+3)(3x+2)(x+2).\end{aligned}$$

$$\begin{aligned}&x^3+(a+b)x^2+(ab+a^2)x+a^2b\\ &=x^3+bx\\ &\qquad+ax^2+abx\\ &\qquad\qquad+a^2x+a^2b\\ &=(x+b)(x^2+ax+a^2)\end{aligned}$$

64. Expressions which are cubics as regards one letter may be only of the first or second degree as regards another. They should always, for the purposes of factorization, be arranged in powers of that letter which gives them the lowest degree.

Thus, $2x^3+(2a-9)x^2-(9a+6)x+27$ is a cubic in x, but is of the first degree only in a. To factorize, arrange in powers of a.

$$\begin{aligned}&2x^3+(2a-9)x^2-(9a+6)x+27\\ &=ax(2x-9)+2x^3-9x^2-6x+27\\ &=ax(2x-9)+x^2(2x-9)-3(2x-9)\\ &=(2x-9)(ax+x^2-3)\end{aligned}$$

Again, $(x+1)^2(x+a)+a^2$ is a cubic in x, but merely a quadratic in a. Arranging in powers of a.

$$\begin{aligned}&(x+1)^2(x+a)+a^2\\ &=a^2+a(1+x)^2+x(1+x)^2\end{aligned}$$

To factorize this we have merely to choose p and q such that

$$p+q=(x+1)^2$$
$$pq=x(x+1)^2$$

which is easily seen to be satisfied by

$$p=x+1, \qquad q=x(x+1)$$

Hence, the required factors are

$$a+x+1 \text{ and } a+x(x+1)$$

65. The "group" method, which we have hitherto only employed for cubics, may be applied with advantage to quadratics which do not yield readily to guessing, especially to the form ax^2+bx+c discussed on page 67.

Since $\quad lq \, . \, mp=lp \, . \, mq=ac,$

the two numbers lq and mp, whose sum is b, have ac for their product.

Hence, to factorize $12x^2+23x+10$, find two numbers whose sum is 23 and whose product is 12×10. These are easily seen to be 8 and 15.

Then arrange thus,

$$\begin{aligned}12x^2+23x+10 &= 12x^2+8x \\ &\qquad +15x+10 \\ &= 4x(3x+2)+5(3x+2) \\ &= (4x+5)(3x+2)\end{aligned}$$

For further discussion of the factors of ax^2+bx+c see D. E. Smith's *The Teaching of Elementary Mathematics*, pp. 192-195, and Chrystal's *Introduction to Algebra*, pp. 181-183.

66. The student will not at the present stage of his algebraical knowledge be asked to factorize expressions of a degree higher than the third, unless they reduce to some of the cases already examined, or unless some clue is given to help him. A few useful examples are added :—

$$\begin{aligned}a^4+a^2b^2+b^4 &= a^4+2a^2b^2+b^4-a^2b^2 \\ &= (a^2+b^2)^2-(ab)^2 \\ &= (a^2+b^2+ab)(a^2+b^2-ab)\end{aligned}$$

$$\begin{aligned} x^4+4 &= x^4+4x^2+4-4x^2 \\ &= (x^2+2)^2-(2x)^2 \\ &= (x^2+2+2x)(x^2+2-2x) \end{aligned}$$

$$\begin{aligned} x^4+2x^2+9 &= x^4+6x^2+9-4x^2 \\ &= (x^2+3)^2-(2x)^2 \\ &= (x^2+2x+3)(x^2-2x+3) \end{aligned}$$

$$\begin{aligned} x^4-13x^2+36 &= (x^2-4)(x^2-9) \\ &= (x+2)(x-2)(x+3)(x-3) \end{aligned}$$

EXERCISE XXX

Factorize :—

(1) $8x^3+12x^2+6x+1$

(2) $x^3-9x^2+27x-27$

(3) $x^3-6x^2y+12xy^2-8y^3$

(4) $125x^3+75ax^2+15a^2x+a^3$

(5) $a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3$

(6) a^3+64 (7) $64a^3-1$

(8) $125a^3+27b^3$ (9) $8a^3-27b^3c^3$

(10) $(a+b)^3-3(a+b)^2b+3(a+b)b^2-b^3$

(11) $(a+b)^3-3(a+b)^2(b+c)+3(a+b)(b+c)^2-(b+c)^3$

(12) $a^3+8b^3+c^3-6abc$ (13) $a^3+b^3-8c^3+6abc$

(14) $x^3+5x^2+2x+10$ (15) $x^3-2x^2+7x-14$

(16) $3x^3-x^2-6x+2$ (17) $x^3-7x(x-2)-8$

(18) $x^3+9x^2+26x+24$ (19) $x^3-9x^2y+26xy^2-24y^3$

(20) $24x^3-26x^2+9x-1$ (21) $x^3-10x^2+31x-30$

(22) $x^3+(b+c)x^2+(bc+c^2)x+bc^2$

(23) $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$

(24) $1+4x^4$ (25) $a^8+a^4b^4+b^8$

(26) $9x^4+2x^2+1$ (27) x^4-25x^2+144

(28) x^4+4y^4 (29) $64x^4+1$

CHAPTER VI

HIGHEST COMMON FACTORS

67. A **Factor** is said to be "common" to two or more algebraical expressions when it will exactly divide each of them.

The expression of highest dimensions which will divide two or more algebraical expressions is said to be their Highest Common Factor, or H.C.F.

68. The Highest Common Factor of monomial expressions—*i.e.* expressions consisting of a single term—may be found by inspection, as by the definition above it is clearly the product of the letters common to all the expressions, raised each to the lowest power of it which occurs in any of them.

Ex. 1. To find the H.C.F. of $a^3b^3c^3$, $a^2b^4d^2$, $a^4b^5c^2d$.

The letter a occurs in each, and the highest power of it which will divide them all is a^2. The highest power of b which will divide them all is b^3. c does not occur in all, neither does d.

Therefore, the H.C.F. is a^2b^3.

Ex. 2. To find the H.C.F. of $12a^2b^3c^5$, $8a^3b^4c^2$, $10ab^2c^3$.

The number 2 is the H.C.F. of the numerical coefficients, therefore 2 is a factor of H.C.F.

The highest power of a common to all three is a.
The highest power of b common to all three is b^2.
The highest power of c common to all three is c^2.

Therefore, the H.C.F. is $2ab^2c^2$.

In actual practice leave the explanatory steps out, and resolve the numerical coefficients into their prime factors thus :—

$$\begin{aligned} 12a^2b^3c^5 &= 2^2 . 3 . a^2b^3c^5 \\ 8a^3b^4c^2 &= 2^3 . a^3 . b^4c^2 \\ 10ab^2c^3 &= 2 . 5 . ab^2c^3 \end{aligned}$$

69. Where the factors of the expressions given are known, the H.C.F. can be found as before by taking the product of the factors which are common to all the given expressions, each raised to the lowest power in which it occurs in any of them.

Ex. 1. To find the H.C.F. of

$$(x-1)^2(x-2)(x-3)^3, \quad (x-2)^2(x-3)^3(x-1)^2$$
$$\text{and} \quad (x-4)^3(x-3)^2(x-2)$$

The factor $(x-1)$ does not occur in each.

The factor $(x-2)$ occurs in all, and its lowest power is $(x-2)$.

The factor $(x-3)$ occurs in all, and its lowest power is $(x-3)^2$.

The factor $(x-4)$ does not occur in each.

Therefore, the H.C.F. is $(x-2)(x-3)^2$.

Ex. 2. Find the H.C.F. of

$$x^2-5x+6 \quad \text{and} \quad x^2-x-6$$

Splitting up the expression into factors, as in the last chapter, we have

$$\begin{aligned} &x^2-5x+6 \qquad\qquad &x^2-x-6 \\ &=(x-3)(x-2) \qquad\qquad &=(x-3)(x+2) \end{aligned}$$

and the only common factor is $\overline{x-3}$.

Therefore, the H.C.F. required is $\overline{x-3}$.

Ex. 3. Find the H.C.F. of

$$b^4c^2 - b^2c^4 \text{ and } b^4c^3 + b^3c^4$$

The factors of the expressions are

$$b^2c^2(b^2 - c^2), \quad b^3c^3(b + c)$$

$$\textit{i.e.} \quad b^2c^2(b + c)(b - c), \quad b^3c^3(b + c).$$

Both expressions are divisible by $b + c$, and also by b^2c^2.

Therefore, the H.C.F. is $b^2c^2(b + c)$.

Ex. 4. Find the H.C.F. of

$$2(a - b)(b - c), \quad 3(b - a)(b - c)$$

Since $(b - a)$ is the same as $-(a - b)$, we may write

$$3(b - a)(b - c) = -3(a - b)(b - c)$$

Whence we see that the H.C.F. is

$$(a - b)(b - c)$$

Similarly, we might have written $2(a - b)(b - c)$ as $-2(b - a)(b - c)$, and the H.C.F. as $(b - a)(b - c)$, which is the same as $-(a - b)(b - c)$.

EXERCISE XXXI

Find the H.C.F. of :—

(1) 10, 15, 20

(2) 21, 14, 42

(3) 4, 14, 24

(4) abc, bcd, $a^2b^2c^2$

(5) x^2y, y^2x, xyz

(6) xy^2z, xyz^2, x^2yz

(7) $2ab$, $4abc$, $50a^2b^2$

(8) $x^3y^3z^3$, $abxy$, $a^3b^3y^2z^4$

(9) $3a^2b^4c^3$, $15a^6b^2c^4$, $21a^3b^7c$

(10) $30a^3y^3$, $150x^3y^2$, $216a^2x^2y^2$

(11) a^6b^2, $4a^4b^4$, $16a^2b^6$

(12) a^3, $3a^2b$, $2ab^2$, a^4, $8a^2b^2$

(13) $a^3b^4c^5$, $a^2b^3c^2$, a^5bc^4

(14) $162x^4y^3z^2$, $18x^2y^3z^3$

(15) $(x - 1)^2(x - 2)$, $(x - 2)^2(x - 1)$, $(x - 1)^3(x - 2)^2(x - 3)$

(16) $a^2b^2(a+1)^2(a-2)^3$, $(a^4b^6(a-3)(a+1)^3(a+2)$,
$a^3b^3(a-2)^2(a+1)$

(17) $(x-1)(x-2)(3-x)$, $(x-2)(1-x)(x-3)$,
$(x-3)(2-x)(x-1)$

(18) $(a-b)(b-c)(c-a)$, $(a-c)(c-b)(b-a)$,
$(a-b)(a-c)(b-c)$

(19) $(y-4)(2y-6)(y-5)$, $(3y-12)(y-3)(2y-10)$,
$(5y-15)(4y-20)(2y-8)$

(20) $24(x-y)(y+z)(y-z)$, $(16y^2-z^2)(x^2-y^2)$,
$8(y-x)(z-y)$

70. The Highest Common Factor of algebraical expressions is often inaccurately called their Greatest Common Measure, or G.C.M.

The inaccuracy may be easily shown by considering the expressions x^2-1 and $\overline{x+1}$. The former is clearly of higher dimensions in x than the latter, but if we give to x a positive value less than 1, say $\frac{1}{5}$, their numerical values are respectively $1\frac{1}{25}$ and $1\frac{1}{5}$, that is to say, the former is numerically less than the latter.

71. If we are unable to find the factors of the given expressions by inspection, the Highest Common Factor may be found by adopting the process in use in Arithmetic for finding the G.C.M. of two numbers.

The rule for finding the Highest Common Factor of two expressions may be stated as follows :—

72. Arrange the two expressions in descending powers of some common letter, and then divide the expression of higher degree in that letter by the other, unless both are of the same degree, when either may be taken as the divisor. Take the remainder, if any, after the first division as a new divisor, and the old divisor as dividend, and repeat the process until there is no remainder. The last divisor will be the H.C.F. of the two expressions.

If any monomial, *i.e.* simple, factor is common to each term of either expression, it should be removed before commencing this process.

If the factor removed be common to *both* expressions, the result obtained must be multiplied by it to obtain the H.C.F.

At any stage in the process, the divisor, dividend, or remainder may be multiplied or divided by any monomial expression.

Ex. 1. Find the H.C.F. of

$$x^2+2x+1 \text{ and } x^3+2x^2+2x+1.$$

$$\begin{array}{r|r|r|l}
x & x^2+2x+1 & x^3+2x^2+2x+1 & x \\
 & x^2+\ \ x & x^3+2x^2+\ \ x & \\
\hline
1 & x+1 & x+1 & \\
 & \underline{x+1} & &
\end{array}$$

Therefore, the H.C.F. is $\overline{x+1}$.

Ex. 2. Find the H.C.F. of

$$8a^3-4a^2-2a+1 \text{ and } 12a^3+4a^2-3a-1$$

Since $8a^3$ will not exactly divide $12a^3$, we multiply the second expression by the factor 2, which, as it is not common to the former expression, will not affect the answer, and we have:—

$$\begin{array}{r|r|r|l}
2a & 8a^3-4a^2-2a+1 & 24a^3+\ \ 8a^2-6a-2 & 3 \\
 & 8a^3-\qquad -2a & 24a^3-12a^2-6a+3 & \\
\hline
-1 & -4a^2\qquad +1 & {}^{*}\,5\overline{)20a^2\qquad -5} & \\
 & \underline{-4a^2\qquad +1} & 4a^2\qquad -1 &
\end{array}$$

Therefore, the H.C.F. is $4a^2-1$.

* The factor 5, which is *not* common to $8a^3-4a^2-2a+1$, may be removed without affecting the answer.

Ex. 3. To find the H.C.F. of

$$2x^5-4x^4+8x^3-12x^2+6x \text{ and } 3x^5-3x^4-6x^3+9x^2-3x$$

we may write the expression as

$$2x(x^4-2x^3+4x^2-6x+3) \text{ and } 3x(x^4-x^3-2x^2+3x-1)$$

The factors 2 and 3 are clearly not common and may be removed. x is common to both, and is consequently a factor of the answer; the other factor is to be found by taking

$$x^4 - 2x^3 + 4x^2 - 6x + 3 \text{ and } x^4 - x^3 - 2x^2 + 3x - 1$$

and proceeding as before.

$$\begin{array}{r|r|r|r}
x & x^4 - 2x^3 + 4x^2 - 6x + 3 & x^4 - x^3 - 2x^2 + 3x - 1 & 1 \\
 & x^4 - 6x^3 + 9x^2 - 4x & x^4 - 2x^3 + 4x^2 - 6x + 3 & \\
\cline{2-3}
4 & 4x^3 - 5x^2 - 2x + 3 & x^3 - 6x^2 + 9x - 4 & x \\
 & 4x^3 - 24x^2 + 36x - 16 & x^3 - 2x^2 + x & \\
\cline{2-3}
 & 19)\underline{19x^2 - 38x + 19} & -4x^2 + 8x - 4 & -4 \\
 & x^2 - 2x + 1 & \underline{-4x^2 + 8x - 4} &
\end{array}$$

The factor 19, which is not common, may be removed. The other factor of the answer is $(x^2 - 2x + 1)$.

Therefore, the H.C.F. is $x(x^2 - 2x + 1)$.

73. The principle on which both the arithmetical method for finding G.C.M. and the algebraical methods for finding H.C.F. *by means of continued divisions* may perhaps be best understood through an illustration with concrete quantities.

The student will easily see that

(1) *If from a sum containing any exact number of half-crowns a smaller sum containing any other exact number of half-crowns were subtracted, the remainder would contain an exact number of half-crowns.*

(2) *If to a sum containing any exact number of half-crowns any other sum containing an exact number of half-crowns were added, the resulting sum would contain an exact number of half-crowns.*

Similarly, for any two concrete quantities which have a common measure, the application to G.C.M. is easy.

Suppose we want to find the G.C.M. of £1 and £1 7s. 6d.

Here the difference, 7s. 6d., must by (1) also contain the G.C.M. an exact number of times.

Hence the G.C.M. is either 7s. 6d. or some aliquot part of 7s. 6d.

Now, it is not 7s. 6d., for 7s. 6d. is not an aliquot part (or measure) of £1.

The largest aliquot part of 7s. 6d. is 2s. 6d. This is a measure of £1, and therefore by (2) of £1 7s. 6d.

It must, therefore, be their G.C.M.

The student should see that by (1) and (2) the G.C.M. of £1 and £1 7s. 6d. is also the G.C.M. of 7s. 6d. and £1, which we can see to be 2s. 6d. He should also be able to see that if we had not been able to find the G.C.M. of 7s. 6d. and £1 by inspection, we might have obtained it as follows :—

Dividing £1 by 7s. 6d., we find that it goes twice and 5s. over. Hence, by (1) the G.C.M. of £1 and 7s. 6d. must be a measure of 5s. It must, therefore, either be 5s. itself, or some aliquot part of 5s. It cannot be 5s. itself, because 5s. is not a measure of 7s. 6d. The largest aliquot part of 5s. is 2s. 6d., which is seen to be a measure of 7s. 6d., and therefore of £1 (which = twice 7s. 6d. + 2s. 6d.). 2s. 6d. must therefore be the G.C.M. of 7s. 6d. and £1, and therefore be the G.C.M. of £1 and £1 7s. 6d.

74. Let us now take the two algebraical expressions $x^3+3x^2-25x-12$ and $2x^3-5x^2-5x-28$. Proceeding as in the arithmetical one, we find that the remainder, when we subtract twice the former from the latter, is $-11x^2+45x-4$.

$$\begin{array}{r}2x^3-\ 5x^2-\ 5x-28\\ 2x^3+\ 6x^2-50x-24\\ \hline -11x^2+45x-\ 4\\ =(x-4)(-11x+1)\end{array}$$

The H.C.F. must either be this expression or some factor of it. Its factors are $x-4$ and $-11x+1$. It is plain that $-11x+1$ does not divide either of the given expressions. Hence, if there is an algebraical common factor, it must be $x-4$. It will be sufficient to try this as a divisor of one of the two given expressions.

Now

$$\begin{aligned} & x^3+3x^2-25x-12 \\ = & x^3-4x^2 \\ & \quad +7x^2-28x \\ & \qquad +3x-12 \\ = & (x-4)(x^2+7x+3) \end{aligned}$$

Hence $x-4$ is a factor of $x^3+3x^2-25x-12$, and as it is also a factor of $-11x^2+45x-4$, it must be a factor of $2x^3-5x^2-5x-28$. But it is as well to check our work. On trial we find

$$\begin{aligned} & 2x^3-5x^2-5x-28 \\ = & 2x^3-8x^2 \\ & \quad +3x^2-12x \\ & \qquad +7x-28 \\ = & (x-4)(2x^2+3x+7) \end{aligned}$$

Hence the H.C.F. is $x-4$.

Similarly, the remainder, when we divide

$$9x^4+11x^2-41x-34 \text{ by } 3x^3-4x^2-8x+8,$$

is $23x^2-17x-58$, which factorizes into $(x-2)(23x+29)$. On trial, either of the given expressions contains $x-2$ as a factor, and as neither contains $23x+29$, $x-2$ must be the required H.C.F.

EXERCISE XXXII

Find the H.C.F. of:—

(1) $x^2-5xy+4y^2$ and $x^3-5x^2y+4y^3$

(2) x^3-5x^2+7x-3 and x^2+x-12

(3) a^3+1 and a^3+6a^2+6a+1

(4) b^4-1 and b^3+3b^2-4

(5) $x^3-8x^2+21x-18$ and $3x^3-16x^2+21x$

(6) $x^8+a^2x^6+ax^2+a^3$ and $x^6-a^4x^2-ax^4+a^5$

(7) $7x^2-12x+5$ and $2x^3+x^2-8x+5$

(8) $y^3-4y^2+9y-10$ and $y^3+2y^2-3y+20$

(9) $x^3-4x^2y+4xy^2-y^3$ and $2x^2-6xy+4y^2$

(10) x^3-3x^2+3x-2 and x^3-4x^2+6x-4

(11) $y^3-3y^2+7y-21$ and $2y^4+19y^2+35$

(12) $x^3-3x^2-10x+24$ and $2bx^3-10bx^2+8bx$

EXERCISE XXXIII

(1) $2x^3-8x^2y+16xy^2-16y^3$ and $8x^2-4xy-24y^2$

(2) $9y^2-3yz-6y+2z$ and $6y^4-4y^3-3yz^2+2z^2$

(3) $y^4-4y^3+8y^2-16y-16$ and $y^4-6y^3+13y^2-12y+4$

(4) $48y^3+8y^2+31y+15$ and $24y^3+22y^2+17y+5$

(5) $3x^3-3cx^2-dx^2-3x^2+3cx+dx$ and $2x^2-2ax+bx-2x+2a-b$

(6) $x^3+(x+1)xy+y^2$ and $x^4-x^2(y^2-y)-y^3$

(7) $11y^4-9ay^3-a^2y^2-a^4$ and $13y^4-10ay^3-2a^2y^2-a^4$

(8) x^4-x^3+x-1 and $x^4-2x^3+3x^2-2x+1$

(9) a^4-b^4 and $a^4+2a^3b+2a^2b^2+2ab^3+b^4$

(10) $x^2+y^2+z^2+2(xy+xz+yz)$ and $x^2-y^2-z^2-2yz$

(11) $15x^4+10x^3y+4x^2y^2+6xy^3-3y^4$ and $6x^3+19x^2y+8xy^2-5y^3$

(12) $x^5-bx^4-2a^2x^3+a^3x^2+a^3bx+2a^2b^2x-2a^3b^2$ and $x^4+a^2b^2-a^2x^2-b^2x^2$

75. To find the H.C.F. of three or more expressions proceed as before to find the H.C.F. of the first two. The factor found will be the factor of highest degree common to those two, and consequently includes all factors which are common to them. The Highest Common Factor of this result, and the third expression, will be also a factor of the first two, and will be the H.C.F. required. If there are more than three ex-

pressions, the H.C.F. of the last result and the fourth expression must be taken, and so on.

N.B.—Always try to split up the given expressions into factors by inspection before trying the longer process.

EXERCISE XXXIV

Find the H.C.F. of:—

(1) $a^2+2ab+b^2$, a^2-b^2, a^3+b^3

(2) $6x^2+4x^2y$, $2ax^4-8bx^2y^2$, $4cx^5+12dx^4y$

(3) $ab+2a^2-3b^2-4bc-ac-c^2$, $2ax+3bx+cx$, $9ac+2a^2-5ab+4c^2+8bc-12b^2$

(4) $a^3+5a^2x+7ax^2+3x^3$, $a^3+3a^2x-ax^2-3x^3$, $a^3+a^2x-5ax^2+3x^3$

(5) x^3-5x^2+7x-3, x^2+x-12, x^3-27

(6) x^3-3x+2, x^3+4x^2-5, x^2-2x+1

(7) $x^3-3x^2-10x+24$, $2ax^3-10ax^2+8ax$, $x^3-12x^2+48x-64$

(8) $a^3-a^2b+3ab^2-3b^3$, $a^2-5ab+4b^2$, a^3-b^3

CHAPTER VII

LOWEST COMMON MULTIPLE

76. A COMMON MULTIPLE of two or more expressions is an expression of which each of the given expressions is a factor.

The expression of lowest dimensions which is exactly divisible by two or more expressions is said to be their Lowest Common Multiple, which is generally written L.C.M.

As in the case of H.C.F., when the expressions are monomial, *i.e.* consist of only one term, their L.C.M. may be found by inspection, as it must clearly be the product of all the letters occurring in any of the expressions, each raised to the highest power to which it occurs in any of them, multiplied by the L.C.M. of their numerical coefficients (if any).

Ex. 1. Find the L.C.M. of a, bc, a^2c, ab^3c.

The highest power of a which occurs is a^2
The highest power of b which occurs is b^3
The highest power of c which occurs is c

and there are no other factors. Therefore, L.C.M. is a^2b^3c.

Ex. 2. Find the L.C.M. of

$$3a^2bc, \quad 5a^5b^4c^2, \quad 13a^2xc^3, \quad 9x^4.$$

Here the L.C.M. of the numerical coefficients is

$$5 \times 13 \times 9$$
$$= 585.$$

The highest power of a is a^5; of b, b^4; of c, c^3; and of x, x^4.

Therefore, the L.C.M. is $585a^5b^4c^3x^4$.

Similarly, if the factors of the expressions whose L.C.M. is required are known, the L.C.M. required will be the product of all the factors which occur, each raised to the highest power in which it is found.

Ex. Find the L.C.M. of (a^2-b^2), $(a-b)^2$, $(a+b)^3$.

The factors are $(a-b)(a+b)$, $(a-b)^2$, $(a+b)^3$.

The highest power of $(a-b)$ is $(a-b)^2$, and of $(a+b)$ $(a+b)^3$.

Therefore, the L.C.M. is $(a-b)^2 \times (a+b)^3$.

EXERCISE XXXV

Find the L.C.M. of:—

(1) a, a^2, a^3
(2) b^3, b^2, b^5, 2
(3) x^4, x^2, x^{11}, 1
(4) $3a$, $4a^4$, 5
(5) $2a^2$, $3a^3$, $4a^4$
(6) $5y^2$, $6y^7$, $2y$
(7) ab, a^3b^2, b^4
(8) $2a$, $5a^3b$, $6b^3$
(9) $3a^4b$, $6a^3b^6$, $5a^3b^2$
(10) 12, $2a^4$, $4a^3b^2$
(11) $5a^2$, $9x$, abc
(12) $2c^2d$, $3bx^2$, $4b^2c$
(13) y^7, $x^4y^{11}z^2$, $3xz^6$
(14) $5c^3z$, $3z^2$, $9ac$
(15) $3x^2y$, $27xy^2z^3$, $6x^3yz^4$
(16) $3a(a+b)$, $5a^2(a+b)^2$
(17) $xy(x-y)$, $yz(x+y)$, $xz(x^2-y^2)$
(18) $2axz(x-z)^3$, $4ax^2(x^2-z^2)$, $3z^3(x+z)^3$
(19) $4y^2z(a^2-b^2)$, $5ab(x^2-z^2)$, $2a^2z^2x^2b^2$
(20) $(a+b)^3$, $3(a+b)^2$, $4(a+b)$, $12bc$

Ex. 3. Find the L.C.M. of

$$(a^2-2a+1),\ (a^2-1),\ \text{and}\ a^2-3a+2$$

Here

$$\begin{aligned} a^2-2a+1 &= (a-1)^2 \\ a^2-1 &= (a+1)(a-1) \\ a^2-3a+2 &= (a-2)(a-1) \end{aligned}$$

The highest power of $(a-1)$ is $(a-1)^2$

„ „ $(a+1)$ „ $a+1$

„ „ $(a-2)$ „ $(a-2)$

Therefore, the L.C.M. is $(a-1)^2 \times (a+1) \times (a-2)$.

EXERCISE XXXVI

Find the L.C.M. of :—

(1) $x^2+5x+6,\ x^2+6x+5,\ x^2+3x+2$

(2) $x^2-7x+6,\ x^2-1,\ x^2-3x-18$

(3) $a^2+8a+7,\ a^2+8a+16,\ a^2b^2c^2$

(4) $a(a-5),\ a^2-4a-5,\ a^2-1$

(5) $3b^2-9b,\ b^2-8b+15,\ 2b^2+18b+40$

(6) $x^2-3x+2,\ x^2-5x+6,\ x^2-4x+3$

(7) $x,\ ax,\ a+x,\ a-x$

(8) $a^2-b^2,\ (a-b)^2,\ a^3+b^3$

(9) $a^2-ax+x^2,\ a^2+ax+x^2,\ a^3-x^3,\ a^3+x^3$

(10) $x^3+1,\ (x+1)^2,\ (x^2-1)$

77. When the factors of the expressions whose L.C.M. is to be found cannot be seen by inspection, their H.C.F. must be found as shown above, and this being known, the other factors can be easily found by division, and then the L.C.M. may be written down as before.

Ex. **Find the L.C.M. of :—**

$$a^4 - a^3 + 8a - 8,\ a^4 + 4a^3 - 8a^2 + 24a$$

$$\begin{array}{r|l|r|l}
 & a^4 - a^3 + 8a - 8 & a^4 + 4a^3 - 8a^2 + 24a & 1 \\
 & 5 & a^4 - a^3 \quad + 8a - 8 & \\
\cline{2-3}
a & 5a^4 - 5a^3 + 40a - 40 & 5a^3 - 8a^2 + 16a + 8 & 5a \\
 & 5a^4 - 8a^3 + 16a^2 + 8a & 5a^3 - 10a^2 + 20a & \\
\cline{2-3}
 & 3a^3 - 16a^2 + 32a - 40 & 2a^2 - 4a + 8 & 2 \\
 & 5 & 2a^2 - 4a + 8 & \\
\cline{2-3}
 & 15a^3 - 80a^2 + 160a - 200 & & \\
 & 15a^3 - 24a^2 + 48a + 24 & & \\
\cline{2-2}
-56 & -56a^2 + 112a - 224 & & \\
\cline{2-2}
 & a^2 - 2a + 4 & &
\end{array}$$

$$\begin{array}{r}
a^2 - 2a + 4)\,a^4 + 4a^3 - 8a^2 + 24a\,(a^2 + 6a \\
a^4 - 2a^3 + 4a^2 \qquad\qquad \\
\hline
6a^3 - 12a^2 + 24a
\end{array}$$

$$\begin{array}{r}
a^2 - 2a + 4)\,a^4 - a^3 + 8a - 8\,(a^2 + a - 2 \\
a^4 - 2a^3 + 4a^2 \qquad\qquad \\
\hline
a^3 - 4a^2 + 8a - 8 \\
a^3 - 2a^2 + 4a \qquad \\
\hline
-2a^2 + 4a - 8 \\
-2a^2 + 4a - 8
\end{array}$$

The H.C.F. is $a^2 - 2a + 4$.

By division the other factors are $a^2 + a - 2$ and $a^2 + 6a$.

Therefore, the L.C.M. is

$$(a^2 - 2a + 4)\,(a^2 + a - 2)\,(a^2 + 6a).$$

EXERCISE XXXVII

Find the L.C.M. of :—

(1) $x^3 - 7x^2 + 16x - 12,\ 3x^3 - 14x^2 + 16x$

(2) $12x^2 - 17ax + 6a^2,\ 9x^2 + 6ax - 8a^2$

(3) $a^3 + 2a^2b - ab^2 - 2b^3,\ a^3 - 2a^2b - ab^2 + 2b^3$

(4) $a^4 - 3a^3b + 4a^2b^2 - 3ab^3 + b^4,\ a^4 + a^3b + ab^3 + b^4$

(5) $x^5 + bx^4 + b^2x^3 + b^3x^2 + b^4x + b^5,\ x^5 - bx^4 + b^2x^3 - b^3x^2 + b^4x - b^5$

(6) $x^2 + y^2 + z^2 + 2(xy + yz + xz),\ x^2 - y^2 - z^2 - 2yz$

(7) $a^2 - 3ab + ac + 2b^2 - 2bc,\ a^2 + b^2 - c^2 - 2ab$

(8) $15a^4 - 9a^3 + 47a^2 - 21a + 28,\ 20a^6 - 12a^5 + 16a^4 - 15a^3 + 14a^2 - 15a + 4$

(9) $2a^4 + 9a^3 + 14a + 3,\ 3a^4 + 15a^3 + 5a^2 + 10a + 2$

(10) $y^4 - 2y^3 + 5y^2 - 4y + 3,\ 2y^4 - y^3 + 6y^2 + 2y + 3$

(11) $x^4 - 1,\ x^3 + 3x^2 - 4,\ x^2 - 2x + 1$

(12) $x^3 - 3x^2 + 3x - 2, x^3 - 4x^2 + 6x - 4, x^3 - 6x^2 + 12x - 8$

CHAPTER VIII

FRACTIONS

78. It cannot be too strongly impressed on beginners that the treatment of Algebraical Fractions is founded on the same principles precisely as that of Arithmetical Fractions. Each set of examples in the former should be preceded by a few examples in the latter.

Thus the meaning of $\frac{a}{b}$ is to be gathered from the meanings already well known of $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{9}$, etc., etc.

The convention that "ab" means the product of a and b prevents the use in Algebra of the notation of the "mixed number"; thus, though in Arithmetic we may write either of the two forms $2\frac{1}{3}$, $2 + \frac{1}{3}$ as an equivalent for $\frac{7}{3}$, we can only use the second form in Algebra, *e.g.* $\frac{7a}{3} = 2a + \frac{a}{3}$, since $2a\frac{a}{3}$ would mean the product of $2a$ and $\frac{a}{3}$.

As in general the values of the numbers a, b, c are not known, the terms *improper fraction* and *proper fraction* have not the same significance as in Arithmetic.

79. In the following sections we have invariably not only worked out an arithmetical example before the corresponding algebraical one, but have prefaced each set of Exercises with a few in Arithmetic to be worked by the students.

To transform a fraction into an integer ± fraction.

Arithmetical Example :—

(1) $\frac{101}{99} = 1 + \frac{2}{99}$

(2) $\frac{99}{101} = 1 - \frac{2}{101}$

Algebraical Example :—

(1) $\frac{a+3}{a+2} = 1 + \frac{1}{a+2}$

(2) $\frac{a+2}{a+3} = 1 - \frac{1}{a+3}$

(3) $\frac{3x+5}{2x-1} = \frac{(2x-1)+(x+6)}{2x-1}$

(4) $\frac{p^2+a+b}{p^2+c+q} = \frac{(p^2+c+q)+(a+b-c-q)}{p^2+c+q}$

EXERCISE XXXVIII

Express as an integer ± a fraction :—

(1) $\frac{21}{19}$

(2) $\frac{73}{74}$

(3) $\frac{115}{26}$

(4) $\frac{100}{113}$

(5) $\frac{3a+1}{3a}$

(6) $\frac{a-1}{a+1}$

(7) $\frac{a+1}{a-1}$

(8) $\frac{2b+2}{b+2}$

(9) $\frac{x}{x-2}$

(10) $\frac{x-2}{x}$

(11) $\frac{b-5}{b-10}$

(12) $\frac{3b+6}{b+1}$

(13) $\frac{21x+17}{7x+5}$

(14) $\frac{21x+17}{6x+6}$

(15) $\frac{21x}{21x-2}$

(16) $\frac{a^2-5}{a^2}$

(17) $\frac{a^2}{a^2+5}$

(18) $\frac{3x^2y^2+2}{x^2y^2+1}$

(19) $\frac{5x+6y}{x+y}$

(20) $\frac{6x+5y}{x+y}$

(21) $\frac{2a^2+3b^2}{a^2-b^2}$

(22) $\frac{x^2y^2+2b}{x^2y^2+c}$

(23) $\frac{xy+a}{xy-b}$

(24) $\frac{3a^2+b-c}{a^2-b+c}$

TO BRING FRACTIONS TO A COMMON DENOMINATOR

Arithmetical Example :—

Bring $\frac{2}{3}$, $\frac{4}{7}$, $\frac{9}{11}$ to a common denominator.

As 3, 7, 11 are prime numbers, *i.e.* have no factor but unity, the least common denominator is seen to be

$$3 \times 7 \times 11$$

$$\frac{2}{3} = \frac{2 \times 7 \times 11}{3 \times 7 \times 11} = \frac{154}{231}$$

$$\frac{4}{7} = \frac{4 \times 11 \times 3}{7 \times 11 \times 3} = \frac{132}{231}$$

$$\frac{9}{11} = \frac{9 \times 7 \times 3}{11 \times 7 \times 3} = \frac{189}{231}$$

Algebraical Examples :—

(1) Bring $\frac{a}{3}, \frac{b}{7}, \frac{c}{11}$ to a common denominator.

As above, the least common denominator is seen to be $3 \times 7 \times 11$

$$\frac{a}{3} = \frac{a \times 7 \times 11}{3 \times 7 \times 11} = \frac{77a}{231} \quad \Bigg| \quad \frac{b}{7} = \frac{b \times 11 \times 3}{7 \times 11 \times 3} = \frac{33b}{231} \quad \Bigg| \quad \frac{c}{11} = \frac{c \times 7 \times 3}{11 \times 7 \times 3} = \frac{21c}{231}$$

(2) Bring $\frac{2}{a}, \frac{4}{b}, \frac{9}{c}$ to a common denominator.

As a, b, c have no common factor except 1, the lowest common denominator is $a \times b \times c$.

$$\frac{2}{a} = \frac{2 \times b \times c}{a \times b \times c} = \frac{2bc}{abc} \quad \Bigg| \quad \frac{4}{b} = \frac{4 \times a \times c}{b \times a \times c} = \frac{4ac}{abc} \quad \Bigg| \quad \frac{9}{c} = \frac{9 \times a \times b}{c \times a \times b} = \frac{9ab}{abc}$$

Arithmetical Example :—

Bring $\frac{2}{77}, \frac{4}{55}, \frac{9}{35}$ to a common denominator.

Here the L.C.M. of the denominators is $11 \times 7 \times 5$, *i.e.* 385, which contains 77 *five* times, 55 *seven* times, and 35 *eleven* times.

Then, since *the value of a fraction is not altered by multiplying both its numerator and denominator by the same number*,

$$\frac{2}{77} = \frac{2 \times 5}{77 \times 5} = \frac{10}{385} \quad \Bigg| \quad \frac{4}{55} = \frac{4 \times 7}{55 \times 7} = \frac{28}{385} \quad \Bigg| \quad \frac{9}{35} = \frac{9 \times 11}{35 \times 11} = \frac{99}{385}$$

Algebraical Examples :—

(1) Bring $\frac{a}{77}, \frac{b}{55}, \frac{c}{35}$ to a common denominator.

As before the lowest common denominator is 385, and

$$\begin{array}{rl|rl|rl}
\frac{a}{77} = & \frac{a \times 5}{77 \times 5} & \frac{b}{55} = & \frac{b \times 7}{55 \times 7} & \frac{c}{35} = & \frac{c \times 11}{35 \times 11} \\
= & \frac{5a}{385} & = & \frac{7b}{385} & = & \frac{11c}{385}
\end{array}$$

(2) Bring $\frac{7}{ab}, \frac{5}{bc}, \frac{11}{ac}$ to a common denominator.

Here the L.C.M. of the denominators is abc.

$$\begin{array}{rl|rl|rl}
\frac{7}{ab} = & \frac{7 \times c}{ab \times c} & \frac{5}{bc} = & \frac{5 \times a}{bc \times a} & \frac{11}{ac} = & \frac{11 \times b}{ac \times b} \\
= & \frac{7c}{abc} & = & \frac{5a}{abc} & = & \frac{11b}{abc}
\end{array}$$

(3) Bring $\frac{x}{y^2z}, \frac{y}{z^2x}, \frac{z}{x^2y}$ to a common denominator.

The L.C.M. of denominators is $x^2 \times y^2 \times z^2$

$$\begin{array}{rl|rl|rl}
\frac{x}{y^2z} = & \frac{x \times x^2z}{y^2z \times x^2z} & \frac{y}{z^2x} = & \frac{y \times xy^2}{z^2x \times xy^2} & \frac{z}{x^2y} = & \frac{z \times yz^2}{x^2y \times yz^2} \\
= & \frac{x^3z}{x^2y^2z^2} & = & \frac{xy^3}{x^2y^2z^2} & = & \frac{yz^3}{x^2y^2z^2}
\end{array}$$

(4) Bring $\frac{x}{(a+b)}$, $\frac{y}{a-b}$, $\frac{z}{a^2-b^2}$ to a common denominator.

The L.C.M. of the denominators is $(a+b)(a-b)$
$= a^2-b^2$

$$\frac{x}{a+b} = \frac{x \times (a-b)}{(a+b)(a-b)} = \frac{x(a-b)}{a^2-b^2}$$

$$\frac{y}{(a-b)} = \frac{y \times (a+b)}{(a-b)(a+b)} = \frac{y(a+b)}{a^2-b^2}$$

$$\frac{z}{a^2-b^2} = \frac{z}{a^2-b^2}$$

(5) Bring $\frac{x}{4a^3b(a+b)}$, $\frac{y}{3ab^3(a-b)}$ to a common denominator.

The L.C.M. of the denominators is $12a^3b^3(a+b)(a-b)$
$= 12a^3b^3(a^2-b^2)$

$$\frac{x}{4a^3b(a+b)} = \frac{x \times 3b^2(a-b)}{4a^3b(a+b) \times 3b^2(a-b)} = \frac{3b^2(a-b)x}{12a^3b^3(a^2-b^2)}$$

$$\frac{y}{3ab^3(a-b)} = \frac{y \times 4a^2(a+b)}{3ab^3(a-b) \times 4a^2(a+b)} = \frac{4a^2(a+b)y}{12a^3b^3(a^2-b^2)}$$

EXERCISE XXXIX

Bring to a common denominator :—

(1) $\frac{4}{5}$, $\frac{9}{11}$, $\frac{6}{7}$, $\frac{2}{9}$

(2) $\frac{8}{13}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{2}{11}$

(3) $\frac{3}{10}$, $\frac{2}{15}$, $\frac{1}{8}$, $\frac{7}{12}$

(4) $\frac{xy}{4}$, $\frac{xb}{7}$, $\frac{xc}{11}$

(5) $\frac{ac}{14}, \frac{ba}{9}, \frac{bc}{21}$

(6) $\frac{a}{5}, \frac{b}{11}, \frac{c}{7}$

(7) $\frac{4}{x}, \frac{11}{y}, \frac{5}{a}$

(8) $\frac{3}{c}, \frac{17}{b}, \frac{6}{d}$

(9) $\frac{5}{xy}, \frac{10}{ab}, \frac{13}{pq}$

(10) $\frac{29}{p}, \frac{13}{q}, \frac{7}{r}$

(11) $\frac{2}{ab}, \frac{3}{bc}, \frac{4}{ca}$

(12) $\frac{11}{ax}, \frac{7}{by}, \frac{3}{ab}$

(13) $\frac{2a}{bcd}, \frac{3b}{cda}, \frac{4c}{dab}$

(14) $\frac{5}{x^2y}, \frac{6}{xy^2}, \frac{7}{xyz}$

(15) $\frac{a}{x^2y^2}, \frac{b}{y^2z^2}, \frac{c}{z^2x^2}$

(16) $\frac{x}{y^2z}, \frac{y^2}{z^2x}, \frac{z}{x^2y}$

(17) $\frac{1}{2x}, \frac{3}{4xy}, \frac{5}{6y}$

(18) $\frac{a}{3b}, \frac{x}{2y}, \frac{z}{6b^2y^2}$

(19) $\frac{a}{x+y}, \frac{b}{x-y}, \frac{c}{x^2-y^2}$

(20) $\frac{b}{3(a+b)}, \frac{c}{5(a-b)}, \frac{a}{10(a^2-b^2)}$

(21) $\frac{x}{a^2+ab}, \frac{y}{b^2+ab}, \frac{z}{ab}$

(22) $\frac{p}{a^2b^2(x+y)}, \frac{q}{b^2a(x-y)}$

(23) $\frac{1}{2}, \frac{p}{q(q+1)}, \frac{a}{b(a+c)}, \frac{x}{2bq(1+q)}, \frac{1}{x(a+c)}$

ADDITION AND SUBTRACTION OF FRACTIONS

81. If the fractions to be added or subtracted have a common denominator, their sum is found by adding or subtracting the numerators, and placing the results over the common denominator.

Ex. 1. $\frac{3}{16}+\frac{9}{16}+\frac{2}{16}-\frac{1}{16}=\frac{3+9+2-1}{16}=\frac{14-1}{16}=\frac{13}{16}$

Ex. 2. $\frac{2a}{b}+\frac{3d}{b}-\frac{a}{b}=\frac{2a+3d-a}{b}=\frac{a+3d}{b}$

Ex. 3. $\frac{4}{a+b}-\frac{2b}{a+b}+\frac{b-2}{a+b}=\frac{4-2b+b-2}{a+b}=\frac{2-b}{a+b}$

If the fractions have not a common denominator, they must be reduced to a common denominator, and then proceed as before.

Ex. 1. $\frac{3}{4}+\frac{5}{12}-\frac{2}{3}$

Here the L.C.M. of 3, 4, and 12 is 12.

Therefore, reducing to a common denominator 12, we have

$$\begin{aligned}\frac{3\times 3}{12}+\frac{5}{12}-\frac{2\times 4}{12} &= \frac{9}{12}+\frac{5}{12}-\frac{8}{12}\\ &= \frac{9+5-8}{12}=\frac{14-8}{12}=\frac{6}{12}\\ &= \frac{1}{2}\end{aligned}$$

Ex. 2. $\frac{a}{b}+\frac{c}{d}-\frac{e}{f}$

L.C.M. of denominators is bdf.

Therefore, transforming the fractions to others, with this as denominator, we have

$$\begin{aligned}&\frac{a\times df}{bdf}+\frac{c\times bf}{bdf}-\frac{e\times bd}{bdf}\\ &=\frac{adf}{bdf}+\frac{bcf}{bdf}-\frac{bde}{bdf}\\ &=\frac{adf+bcf-bde}{bdf}\end{aligned}$$

Ex. 3. $\dfrac{2}{p+q} + \dfrac{1}{p-q} - \dfrac{4}{(p^2+q^2)}$

The L.C.M. of the denominators is $(p+q)(p-q)(p^2-q^2)$

Therefore, we have

$$\frac{2(p-q)(p^2+q^2)}{(p+q)(p-q)(p^2+q^2)}+\frac{(p+q)(p^2+q^2)}{(p-q)(p+q)(p^2+q^2)}-\frac{4(p+q)(p-q)}{(p-q)(p^2+q^2)(p+q)}$$

$$=\frac{2p^3+2pq^2-2p^2q-2q^3+p^3+pq^2+p^2q+q^3-4p^2+4q^2}{(p+q)(p-q)(p^2+q^2)}$$

$$=\frac{3p^3-p^2q-4p^2+3pq^2+4q^2-2q^3}{p^4-q^4}$$

Ex. 4. $a + \dfrac{b}{c} + \dfrac{c}{a} - \dfrac{c}{b}$

Considering a as a fraction whose denominator is 1, we have as our common denominator abc, and the fraction becomes

$$\frac{a^2bc}{abc} + \frac{ab^2}{abc} + \frac{bc^2}{abc} - \frac{ac^2}{abc}$$

$$=\frac{a^2bc+ab^2-ac^2+bc^2}{abc}$$

Ex. 5. $\dfrac{p+q}{p-q} - \dfrac{p-q}{p+q}$

Here the common denominator is $(p+q)(p-q)$ or p^2-q^2, and we have

$$\frac{(p+q)(p+q)}{(p-q)(p+q)} - \frac{(p-q)(p-q)}{(p+q)(p-q)}$$

$$=\frac{p^2+2pq+q^2 \ - \ (p^2-2pq+q^2)}{p^2-q^2}$$

$$=\frac{p^2+2pq+q^2-p^2+2pq-q^2}{p^2-q^2}$$

$$=\frac{4pq}{p^2-q^2}$$

EXERCISE XL

Add together :—

(1) $\frac{a}{5}, \frac{2a}{5}, \frac{3a}{5}$

(2) $\frac{3y}{10}, \frac{6y}{10}, \frac{9y}{10}$

(3) $\frac{4pq}{3}, \frac{pq}{12}$

(4) $\frac{3}{a}, \frac{1}{2a}, \frac{3}{4a}$

(5) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$

(6) $\frac{2}{x}, \frac{3}{2x}, \frac{4}{3x}$

(7) $\frac{2a+b}{5}, \frac{3a-b}{5}$

(8) $\frac{a-2b}{10}, \frac{4a+b}{5}$

(9) $\frac{y}{4}, \frac{2y+3}{12}, \frac{y-8}{3}$

(10) $\frac{2x+4}{3}, \frac{3x-7}{9}, \frac{4x-41}{18}$

Find the value of :—

(11) $\frac{a+b}{2} + \frac{a-b}{2}$

(12) $\frac{a+x}{a-x} + \frac{a-x}{a+x}$

(13) $1 + \frac{x-y}{x+y}$

(14) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$

(15) $\frac{p}{p+q} + \frac{q}{p-q}$

(16) $a-b+\frac{4ab}{a-b}$

(17) $a-b+\frac{b^2}{a+b}$

(18) $p+q+\frac{p-q}{q^2}$

(19) $y+x+\frac{x}{y^2-xy}$

(20) $2y-x+\frac{y^2-2xy+x^2}{x}$

EXERCISE XLI

Find the value of :—

(1) $\frac{y+3}{13y} - \frac{y-6}{26y} + \frac{y+3}{39y}$

(2) $\frac{a+b}{2a} + \frac{b+c}{2b} + \frac{a+c}{2c}$

(3) $\frac{2a+8}{a} - \frac{a+4}{2a} - \frac{19}{4a^2}$

(4) $\dfrac{2}{x+1} + \dfrac{3}{2x+2} + 1$

(5) $\dfrac{p-q}{p} + \dfrac{p+q}{q} - \dfrac{p^2-q^2}{2pq}$

(6) $\dfrac{4}{5a} - \dfrac{1}{2a} + \dfrac{3}{4a}$

(7) $\dfrac{a^2-ab}{ab} + \dfrac{b^2-bc}{bc} + \dfrac{c^2-ca}{ca}$

(8) $\dfrac{p+q}{pq} + \dfrac{p-q}{2} - p$

(9) $\dfrac{5x+4}{9} - \dfrac{10x+5}{18} + 2$

(10) $\dfrac{a}{b} - \dfrac{a+b}{ab} + b$

EXERCISE XLII

(1) $\dfrac{3x+2}{x} + \dfrac{4x+3}{2x} + \dfrac{5x+4}{x^2}$

(2) $\dfrac{x}{x^2-y^2} + \dfrac{y}{x+y} + \dfrac{1}{x-y}$

(3) $\dfrac{a}{(a+b)} + \dfrac{a-2b}{(a+b)^2} + 3$

(4) $\dfrac{2x}{y} + \dfrac{3x^2}{5} + \dfrac{2y}{x} + \dfrac{1}{4}$

(5) $\dfrac{5x-3}{x+1} - \dfrac{3x+2}{x-1} - 2$

(6) $\dfrac{5y-3}{y+1} - \dfrac{2y^2-13y+1}{y^2-1} - 3$

(7) $\dfrac{3}{1+x^3} - \dfrac{x-2}{1-x+x^2} + \dfrac{4}{1+x}$

(8) $\dfrac{a}{a+b} - \dfrac{a}{a-b} + \dfrac{b}{a-b}$

(9) $\dfrac{x^2+y^2}{x^2-y^2} + \dfrac{x}{x+y} + \dfrac{y}{y-x}$

(10) $\dfrac{b}{b-c} + \dfrac{b}{b+c} + \dfrac{2b^2}{b^2+c^2}$

82. Care should be taken in finding the value of expressions such as :

$$\frac{2x-1}{3} + \frac{x-5}{6} - \frac{x-4}{4}$$

to first express the numerators in brackets, and then remove them as in the example below. This will avoid a possible mistake in sign.

Here the L.C.M. of the denominators is 12.

Hence, $\dfrac{2x-1}{3} + \dfrac{x-5}{6} - \dfrac{x-4}{4}$

$$= \frac{(2x-1)\times 4}{3\times 4} + \frac{(x-5)\times 2}{6\times 2} - \frac{(x-4)\times 3}{4\times 3}$$

$$= \frac{(8x-4)}{12} + \frac{(2x-10)}{12} - \frac{(3x-12)}{12}$$

$$= \frac{8x-4+2x-10-3x+12}{12}$$

$$= \frac{7x-2}{12}$$

The intermediate steps, after a little practice, may be left out, and the working will then read as follows:

$$\frac{2x-1}{3} + \frac{x-5}{6} - \frac{x-4}{4} = \frac{4(2x-1)+2(x-5)-3(x-4)}{12}$$

$$= \frac{8x-4+2x-10-3x+12}{12}$$

$$= \frac{7x-2}{12}$$

Ex. $\dfrac{2p+3q}{p-2q} - \dfrac{2p-3q}{p+q} + 3.$

The L.C.M. of the denominators is $(p-2q)(p+q)$.

The expression

$$=\frac{(2p+3q)(p+q)-(2p-3q)(p-2q)+3(p+q)(p-2q)}{(p-2q)(p+q)}$$

$$=\frac{2p^2+3pq+2pq+3q^2-(2p^2-3pq-4pq+6q^2)+3(p^2-2pq+pq-2q^2)}{(p-2q)(p+q)}$$

$$=\frac{2p^2+3pq+2pq+3q^2-2p^2+3pq+4pq-6q^2+3p^2-6pq+3pq-6q^2}{(p-2q)(p+q)}$$

$$=\frac{3p^2+9pq-9q^2}{(p-2q)(p+q)}$$

This result may also be written

$$\frac{3(p^2+3pq-3q^2)}{p^2-pq-2q^2}$$

Note that all the multiplication is written down in full as it is done, and like terms are collected afterwards. Thus, in the products of $(2p-3q)$ and $p-2q$ each term of the former expression is multiplied by p, and then each is multiplied by $-2q$ and the products written down in full. This method avoids the necessity of doing the rough work separately, and the whole of the working is shown in the sum itself.

MULTIPLICATION AND DIVISION

83. *To multiply fractions,* multiply the numerators together for a new numerator and the denominators together for a new denominator. The former result over the latter will be the required product.

For the purposes of this rule, a whole number may be regarded as a fraction whose denominator is 1.

Example, $a = \dfrac{a}{1}$

Ex. 1. Simplify

$$\frac{3a^2}{5} \times \frac{7}{2b^2} = \frac{3a^2 \times 7}{5 \times 2b^2} = \frac{21a^2}{10b^2}$$

Should the result thus obtained contain any factors common to both numerator and denominator, this factor may be cancelled out of each, for as in Arithmetic, *the value of a fraction is not altered by dividing both its numerator and denominator by the same number.*

Ex. 2. Simplify

$$\frac{(a+b)^2}{27ab} \times \frac{15b^2}{a^2-b^2} = \frac{(a+b)(a+b) \times 15b^2}{27ab(a+b)(a-b)}$$

Cancelling the factor $(a+b)$, which is common to both numerator and denominator, and also the factor $3b$, we have

$$\frac{(a+b) \times 5b}{9a(a-b)} = \frac{5b(a+b)}{9a(a-b)} \text{ or } \frac{5ab+5b^2}{9a^2-9ab}$$

84. To divide one fraction by another, invert the divisor and proceed as in multiplication. Should the divisor be a whole number, such as a, regarding it as equivalent to $\frac{a}{1}$, when inverted it becomes $\frac{1}{a}$.

Ex. 1. Simplify

$$\frac{3a^2b}{15} \div \frac{6ab^2}{25} = \frac{3a^2b}{15} \times \frac{25}{6ab^2}$$
$$= \frac{75a^2b}{90ab^2}$$
$$= \frac{5a}{6b}$$

Dividing both numerator and denominator by the common factor, $15ab$.

Ex. 2. Simplify $\dfrac{40x^2-16}{5y^2-20} \div \dfrac{3x+2}{2y+4}$

$$\begin{aligned}
\frac{40x^2-16}{5y^2-20} \div \frac{3x+2}{2y+4} &= \frac{8(5x^2-2)}{5(y^2-4)} \times \frac{2(y+2)}{3x+2} \\
&= \frac{8(5x^2-2)}{5(y+2)(y-2)} \times \frac{2(y+2)}{3x+2} \\
&= \frac{16(5x^2-2)}{5(y-2)(3x+2)} \quad (\text{cancelling } \overline{y+2}) \\
&= \frac{80x^2-32}{15xy+10y-30x-20}
\end{aligned}$$

Ex. 3.

$$\begin{aligned}
&\frac{4ax}{3by} \div \frac{c^2-x^2}{a^2-x^2} \times \frac{bc+bx}{a^2-ax} \\
&= \frac{4ax}{3by} \times \frac{a^2-x^2}{c^2-x^2} \times \frac{bc+bx}{a^2-ax} \\
&= \frac{4ax}{3by} \times \frac{(a+x)(a-x)}{(c+x)(c-x)} \times \frac{b(c+x)}{a(a-x)} \\
&= \frac{4x(a+x)}{3y(c-x)} = \frac{4ax+4x^2}{3cy-3xy}
\end{aligned}$$

EXERCISE XLIII

Simplify (leaving the results *in factors*):—

(1) $\dfrac{a}{bx} \times \dfrac{cx}{d}$ (2) $\dfrac{ax}{1+x} \times \dfrac{bx}{1-x}$

(3) $\dfrac{2y}{a} \times \dfrac{3b}{ac} \times \dfrac{5ac}{2b}$ (4) $\dfrac{a-bx}{mx-n} \times \dfrac{a+bx}{mx+n}$

(5) $\dfrac{a^2-x^2}{p+q} \times \dfrac{p^2-q^2}{ax+x^2} \times \dfrac{a^2}{a-x}$

(6) $\dfrac{4x^2-a^2}{8x^3+a^3} \times \dfrac{2x+a}{2x-a}$

(7) $\dfrac{p^4-q^4}{p^2-2pq+q^2} \times \dfrac{p-q}{p^2+pq}$

(8) $\dfrac{x+1}{6} \times \dfrac{2}{3x}$ (9) $\dfrac{2ax+x^2}{c^2+x^2} \times \dfrac{c+x}{x}$

(10) $\dfrac{p^2q^2-1}{a^2-b^2} \times \dfrac{a-b}{pq+1}$ (11) $\dfrac{p^2-q^2}{(p+q)^2} \times \dfrac{p+q}{p^2+q^2}$

(12) $\dfrac{4x^2-4a^2}{2ax+2a^2} \times \dfrac{2a}{2x-2a}$

(13) $\dfrac{a+x}{a} \times \dfrac{a-x}{x} \times \dfrac{a^2-x^2}{a^2+x^2}$

(14) $\dfrac{x^2-y^2}{x} \times \dfrac{x}{x+y} \times \dfrac{1}{x-y}$

(15) $\dfrac{a^2-x^2}{ax} \times \dfrac{a^2-x^2}{a^2+x^2}$ (16) $\dfrac{abx+b^2}{abx-a^2} \times \dfrac{bx-a}{b}$

(17) $\dfrac{(a-b+c)}{c} \times \dfrac{ac-c^2}{b}$ (18) $\dfrac{(a-x)^2}{(a+x)^2} \times \dfrac{a+x}{a-x}$

EXERCISE XLIV

(1) $\dfrac{a+b}{2} \div \dfrac{a-b}{2}$ (2) $\dfrac{3xy}{5} \div \dfrac{9x^2}{15}$

(3) $\dfrac{21pqr}{16a^2} \div \dfrac{34q^2}{4a}$ (4) $\dfrac{a^2}{c} \div \dfrac{b^2}{2c^2}$

(5) $\dfrac{ax}{by} \div \dfrac{cx}{py}$ (6) $\dfrac{6ax^2}{b^4} \div \dfrac{3a^2x}{2b^3}$

(7) $\dfrac{2b(1-4x^2)}{c^2} \div \dfrac{(1-2x)(1+2x)^2}{b^3}$

(8) $3a+3b \div \dfrac{a-b}{3a+6b}$ (9) $\dfrac{a-b}{3a+6b} \div \dfrac{a^2-b^2}{a+2b}$

(10) $\dfrac{p^2-q^2}{p^4-4q^4} \div \dfrac{a-b}{p^2-2q^2}$ (11) $\dfrac{x}{x^2+y^2} \div \dfrac{(x-y)^2}{x^4-y^4}$

(12) $\frac{3b+y}{b^2+by+y^2} \div \frac{b-y}{y}$

(13) $\frac{a^3}{b^3x^3} \div \left(\frac{a}{bx}\right)^3$ (14) $\frac{x^2-49}{x^2-9} \div \frac{x+7}{x-3}$

(15) $\frac{p^2q^2+6pq}{16p^2-1} \div \frac{pq+6}{2p-5}$

(16) $\frac{a-3}{4b-3} \div \frac{a^2-6a+9}{16b^2-9}$

(17) $\frac{2x+b}{3bx+2b^2} \div \frac{4x^2-b^2}{9x^2-4b^2}$

(18) $\frac{(a+b)a^2}{b^2} \div \frac{a^2}{x-y} \div \frac{x^2-y^2}{a-b}$

CHAPTER IX

EQUATIONS

85. **Statements** asserting that an algebraical expression is equal to a number, or to another algebraical expression, are called equations. They are of two classes.

(1) Those which are true for all numerical values of the letters used, such as

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Which are called *Identical Equations.*

(2) Those which are only true for certain definite numerical values, such as

$$\text{(i.)}\quad x+1 = 7$$
$$\text{(ii.)}\quad 8x - x^2 = 12$$

(i.) being only true when $x = 6$.
(ii.) being only true when $x = 2$ or 6.

Which are called *Conditional Equations.* Some writers restrict the term Equation to the second class, and call the others *Identities.*

86. An equation like (i.), which contains no power of x higher than the first, is called a *Simple Equation.* When a simple equation is given us, and we are told to *solve* it, we are required to find what value of x

will make the statement true. This value is called the *root* of the given equation.

87. The various operations performed in the solution of simple equations depend on the following axioms or self-evident propositions:—

(1) If the same quantity be added to two equal quantities the results will be equal.
(2) If the same quantity be subtracted from two equal quantities the remainders will be equal.
(3) If two equal quantities be multiplied by the same quantity the products are equal.
(4) If two equal quantities be divided by the same quantity the quotients are equal.

The teacher should at intervals insist on his pupils explaining each line of their work in equations by reference to the four axioms above, and point out that unless a step can be justified by them it is incorrect. This is best done in class.

By applying these four propositions to the equation to be solved, the value of the unknown quantity can in general be found.

88. *Ex.* 1. Solve $x+5 = 6$

Here we are told that if 5 be added to x the sum is 6.

That is, that x by itself is five less than 6; or, that x represents the difference between 6 and 5, or $6-5$, *i.e.* 1.

Hence, to solve an equation of the form $x+p=q$, we must subtract p from both sides of the equation,

$$\text{i.e.}\quad x+p-p=q-p$$
$$\text{i.e.}\quad x=q-p$$

89. From the methods adopted in the examples given, the following rule for the solution of a simple equation may be found.

Transpose all the terms which contain the unknown quantity to one side of the equation, and all the other terms to the other side, changing the sign of any term so transposed.

Next combine all the terms which contain the unknown quantity into one term, and divide both sides by the coefficient thus obtained. The quotient is the required root of the equation.

Ex. 2. Solve $3x-11=2x+5$

In order to get all the terms containing x on one side of the equation, we subtract $2x$ from each side, which leaves no term containing x on the right hand side.

Thus, $3x-11-2x=2x+5-2x$

or $x-11=5$

To get all the terms which do not contain x on the other side of the equation, -11 is subtracted from each.

Thus,
$$\begin{aligned} x-11-(-11)&=5-(-11)\\ x-11+11&=5+11\\ x&=16 \end{aligned}$$

In actual practice the working would be shown as follows :—

$$\begin{aligned} 3x-11&=2x+5\\ 3x-2x&=5+11\\ x&=16 \end{aligned}$$

Ex. 3.
$$\begin{aligned} b(x-b)&=2bc-cx+c^2\\ bx-b^2&=2bc-cx+c^2 \end{aligned}$$

Collecting all terms containing x on the left-hand side and those which do not contain x on the right-hand side,

$$\begin{aligned} bx+cx&=b^2+2bc+c^2\\ x(b+c)&=(b+c)(b+c) \end{aligned}$$

Therefore, $$x=\frac{(b+c)(b+c)}{(b+c)}=b+c$$

Ex. 4. $3(x+3)^2+5(x+5)^2=8(x+8)^2$

$$3(x^2+6x+9)+5(x^2+10x+25)=8(x^2+16x+64)$$
$$3x^2+18x+27+5x^2+50x+125=8x^2+128x+512$$
$$+18x+50x-128x=512-125-27$$
$$-60x=360$$
$$60x=-360$$
$$x=\frac{-360}{60}=-6.$$

90. Many simple equations appear, as in the example above, to contain powers higher than the first, but on collecting them on the same side of the equation, they will be found to cancel.

EXERCISE XLV

(1) $2x+3=x+5$

(2) $3x+5=x+9$

(3) $2x-7=x+6$

(4) $11x-5=4x+9$

(5) $16x+2=12x+6$

(6) $7x-12=5x+6$

(7) $7x-3=5x+13$

(8) $3x+5=10x-16$

(9) $45x-72=60+x$

(10) $12x-6x+4x=84+3x$

(11) $4x-2=3x+4$

(12) $19x+13=59-4x$

(13) $3x+7=100$

(14) $4x+36=5x+34$

(15) $4x-12+3x+1=2x+4$

(16) $55-2x=8x-35$

(17) $98+24x-48=6x+68$

(18) $11x+21=x-9$

(19) $5x-9=11x-33$

(20) $2x+15=11x+6$

(21) $13x-7=5x+9$

(22) $15x-40=8x+100$

(23) $ax+b=bx+a$

(24) $(b+c)x-(a+b)=(c-a)$

(25) $12(x-2)-4(x-3)+3(x-4)=48$

(26) $20(x-3)-15(x-8)+12(x-5)=0$

(27) $b(x-b)=c(x-c)$

(28) $(x+c)(x+d)-(x-c)(x-d)=(c+d)^2$

(29) $p(2x-p)+q(2x-q)=2pq$

(30) $2(x+2)^2+3(x+3)^2=5(x+1)^2+46$

91. Equations which involve fractions should be cleared of fractions by multiplying *both* sides by the L.C.M. of the denominators, and then proceed as usual.

Ex. $$\frac{12-4x}{10}-\frac{2x+5}{5}=3+\frac{7x+60}{2}-50$$

The L.C.M. of the denominators is 10.

Therefore, multiplying both sides of the equation by 10, we have

$$\begin{aligned} 12-4x-2(2x+5)&=30+5(7x+60)-500\\ 12-4x-4x-10&=30+35x+300-500\\ -4x-4x-35x&=30+300-500-12+10\\ -43x&=340-512\\ -43x&=-172\\ 43x&=172\\ x&=4 \end{aligned}$$

EXERCISE XLVI

(1) $\frac{x}{6}-\frac{x}{4}+10=\frac{x}{3}-\frac{x}{2}+1$

(2) $\frac{5x+3}{4}+7=\frac{4x-10}{10}+10$

(3) $\frac{2x-5}{4}-\frac{7x+10}{3}=16-\frac{12x-10}{5}$

(4) $13\frac{3}{4}-\frac{x}{2}=2x-8\frac{3}{4}$

(5) $12\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3x}{4}-5\frac{3}{8}$

(6) $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$

(7) $21 + \dfrac{3x - 11}{16} = \dfrac{5x - 5}{8} + \dfrac{97 - 7x}{2}$

(8) $15x - 24 = 20 + \dfrac{x}{3}$

(9) $x - \dfrac{x}{2} + \dfrac{x}{3} - \dfrac{x}{4} = 7$

(10) $\dfrac{x}{2} - \dfrac{5x + 4}{3} = \dfrac{4x + 9}{3}$

EXERCISE XLVII

(1) $\dfrac{x + 1}{2} + \dfrac{x + 2}{3} = 14 + \dfrac{5 - x}{4}$

(2) $\dfrac{x + 1}{2} + \dfrac{x + 2}{3} + \dfrac{x + 3}{4} = 16$

(3) $\dfrac{x + 6}{4} + \dfrac{16 - 3x}{12} = \dfrac{x + 9}{6}$

(4) $\dfrac{x + 1}{2} + \dfrac{x + 2}{3} = 16 - \dfrac{5x + 1}{4}$

(5) $\dfrac{x - 2}{2} + \dfrac{x}{3} = 20 - \dfrac{x - 6}{2}$

(6) $\dfrac{x - 7}{11} - \dfrac{3x - 5}{7} + \dfrac{125}{77} = 2x - 17$

(7) $\dfrac{5x - 7}{3} - \dfrac{3x - 2}{7} = \dfrac{x - 5}{4}$

(8) $\dfrac{x + 3}{2} - \dfrac{11 - x}{5} = \dfrac{3x - 1}{20} + 3\tfrac{1}{5}$

(9) $x - \dfrac{x - 2}{3} = 5\tfrac{3}{4} - \dfrac{x + 10}{5} + \dfrac{x}{4}$

(10) $\dfrac{9x + 7}{2} - x + \dfrac{x - 2}{7} = 36$

EXERCISE XLVIII

(1) $$\frac{3x+7}{14} - \frac{2x-7}{21} + 2\tfrac{3}{4} = \frac{x-4}{4}$$

(2) $$\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}$$

(3) $$\frac{4x-21}{7} + 7\tfrac{3}{4} + \frac{7x-28}{3} = x + 3\tfrac{3}{4} - \frac{9-7x}{8}$$

(4) $$\frac{5x-2}{3} + 3\tfrac{1}{2} - \frac{7x-2}{4} = 1 + \frac{12x-14}{17}$$

(5) $$\frac{x-3\frac{1}{2}}{4} + \frac{5x-37}{8} = 5 + \frac{2x-5}{5}$$

(6) $$\frac{x-1\frac{25}{26}}{2} - \frac{2-6x}{13} = x - \frac{20x-10+3x}{156}$$

(7) $$\frac{17-3x}{5} - \frac{4x+2}{3} = \frac{29-11x}{3}$$

(8) $$\frac{3-7x}{9} + \frac{14x-48}{10} = 2 + \frac{7x-54}{30}.$$

(9) $$\frac{2\frac{1}{2}-x}{2} + \frac{3\frac{1}{4}+x}{4} = \frac{1-x}{8} + 1\tfrac{13}{16}$$

(10) $$\frac{3x-1}{4} + \frac{5x+1}{6} + \frac{2x-11}{9} - \frac{7x-13}{12} = \frac{11x+7}{18} - \frac{499-13x}{36} - 14$$

EXERCISE XLIX

Solve :—

(1) $20-8x+8=10x+10$

(2) $6x-24+9x-30-18x+66=0$

(3) $35x+20-15x=60+x-50+48$

(4) $\frac{x}{2} - 3\frac{1}{2} = 3\frac{1}{2} + 7\frac{1}{2}x$

(5) $13\frac{3}{4} - \frac{3x}{2} = 6x - 8\frac{3}{4}$

(6) $30x + \frac{56x}{3} + \frac{220x}{3} = 366$

(7) $x + \frac{2x}{3} - \frac{x}{2} + \frac{2x}{5} = 7\frac{5}{6}$

(8) $\frac{7}{3x} - 4 = \frac{2}{3}$

(9) $\frac{2}{x} + \frac{1}{x} - \frac{2}{3x} = \frac{7}{3}$

(10) $\frac{6x+1}{2} - \frac{4x}{3} = 10 + \frac{2x-1}{6}$

(11) $5x + \frac{5x-2}{3} = \frac{5x-1}{2} + 4$

(12) $\frac{2x+2}{7} + \frac{x}{6} = x-3$

(13) $\frac{3x-5}{4} + \frac{7x-9}{3} = 2x - 2 + \frac{5-x}{2}$

(14) $\frac{x+1}{2} + \frac{1}{1\frac{1}{2}} = \frac{x-1}{3} + \frac{x+4}{3}$

(15) $\frac{x - 7\frac{1}{2}}{2} = \frac{3x - 9}{4} + \frac{27 - 3x}{3}$

(16) $x - 1 + \frac{9 + x}{5} - 5\frac{3}{4} = \frac{x - 3}{3} + \frac{x - 1}{4}$

(17) $4x - \frac{12x - 2}{5} = 3 - \frac{8x - 5}{3}$

(18) $\frac{12x - 14}{42} - \frac{126 - 4x}{21} = \frac{x}{7} + 2$

(19) $10x - \frac{4x - 1}{3} + 1 = 6x + \frac{2x + 2}{2} + 7$

(20) $\frac{x + 5}{4} - \frac{19 - 3x}{12} = \frac{25}{6}$

(21) $\frac{7x - 2}{3} - \frac{12 + 4x}{5} + 6 = \frac{3x + 6}{2}$

(22) $\frac{x + 14}{3} - 5\frac{3}{4} = \frac{2x + 19}{7} - \frac{25 + 5x}{8}$

(23) $\frac{2x + 11}{7} - \frac{9x + 10}{11} = \frac{x - 9}{2}$

(24) $\frac{6x - 1}{7} + \frac{6 - 2x}{4} - \frac{4x - 4}{12} = 2 - \frac{x + 4}{28}$

(25) $\frac{7x + 9}{8} - \frac{3x + 1}{7} = \frac{9x - 13}{4} - \frac{249 - 9x}{14}$

(26) $\frac{3x + 6}{16} + \frac{7x + 3}{15} - \frac{7x + 8}{20} = 2$

(27) $\frac{x}{2} - 4\frac{1}{2} - \frac{12 - 2x}{5\frac{1}{2}} + x = \frac{8\frac{1}{2}}{11}$

(28) $\frac{7x + 12}{23} + \frac{9x + 8}{10} - \frac{x - 8}{5} + \frac{2x - 1}{15} = 23\frac{1}{3}$

(29) $\dfrac{12x+1}{29} - \dfrac{402-18x}{12} = 9 - \dfrac{471-36x}{2}$

(30) $52-5(6x-1)=27$

(31) $4(2x-3)-7(2x-4)=6-2x$

(32) $20(x+1)-50(3x-1)=200-20(4x-1)-(3x-1)50$

(33) $4(5x-8)-64(4-x)-3(12x-16)=96$

(34) $10\left(3x+\dfrac{1}{2}\right) - 18x\left(\dfrac{1}{3x} - \dfrac{1}{3}\right) = 23$

(35) $\dfrac{8x}{5} - \dfrac{5x}{2} + 18 = \dfrac{1}{9}\left(8x+1\right)$

(36) $\dfrac{1}{3}\left(2x-10\right) - \dfrac{1}{11}\left(3x-40\right) = 15 - \dfrac{1}{5}\left(57-x\right)$

(37) $\dfrac{x}{4} - \dfrac{1}{8}\left(8-3x\right) - \dfrac{1}{4}\left(5+3x\right) + \dfrac{11}{4} = 0$

(38) $\dfrac{1}{7}\left(\dfrac{x}{9} - \dfrac{1}{2}\right) - \dfrac{1}{5}\left(\dfrac{2}{3} - \dfrac{x}{9}\right) = 1\tfrac{13}{30}$

(39) $\dfrac{1}{4}\left(4+\dfrac{x}{2}\right) - \dfrac{1}{7}\left(\dfrac{2x}{3} - \dfrac{1}{3}\right) = \dfrac{31}{28}$

(40) $\dfrac{1}{14}\left(\dfrac{3x}{5}+\dfrac{2}{3}\right) - \dfrac{1}{7}\left(\dfrac{4x}{5}-6\tfrac{2}{3}\right) = \dfrac{1}{2}\left(x-6\right)$

(41) $\dfrac{5x-1}{15} - \dfrac{4}{5} \times \dfrac{7x-117}{39} = \dfrac{4}{5}\left(6+\dfrac{x-13}{39}\right)$

(42) $\dfrac{1}{2}\left(\dfrac{2}{3}x+4\right) - \dfrac{7\frac{1}{2}-x}{3} = \dfrac{x}{2}\left(\dfrac{6}{x}-1\right)$

(43) $3\tfrac{1}{3} \times \left\{28-\left(\dfrac{x}{4}+24\right)\right\} = 3\tfrac{1}{2} \times \left\{2\tfrac{1}{3}+\dfrac{x}{2}\right\}$

(44) $\frac{20x+17}{18} - \frac{24x+2}{22x-8} = \frac{10x-4}{9}$

(45) $\frac{24x+13}{15} - \frac{12x+5}{20x-25} = \frac{8x}{5}$

(46) $\frac{4x}{55-x} - \frac{220-4x}{x} = \frac{165}{x}$

(47) $\frac{5x+6}{x-2} + \frac{2x-6}{2x-4} = 9$

(48) $\frac{36+6x}{x+2} + \frac{68+8x}{x+4} = 14 + \frac{48}{x+2}$

CHAPTER X

PROBLEMS

92. If certain relations are known to exist between two or more unknown quantities, we are able, by expressing these relations in the form of equations, to obtain values for the unknown quantities which will in general satisfy the conditions of the question.

No general rule can be given for the solution of problems of this description, but the method to be adopted is to assume symbols x, y, etc., for the quantities to be found, and then express in algebraical language what is stated in the question. After this the necessary equation or equations can be built up.

A few worked examples will give a better idea of the method.

Ex. 1. Divide the number 9 into two parts, such that twice the greater may exceed five times the less by 4.

Let x be the greater part.

Then $9-x$ is the smaller (since the sum of the parts is 9).

$$2x = \text{twice the greater part.}$$
$$5(9-x) = \text{five times the smaller.}$$

$$\therefore \quad 2x - 5(9-x) = \begin{cases}\text{excess of twice the greater} \\ \quad \text{part over five times the less.}\end{cases}$$

$$\begin{aligned} \therefore \quad 2x - 5(9-x) &= 4 \\ 2x - 45 + 5x &= 4 \\ 7x &= 49 \\ x &= 7 \\ \therefore \quad 9 - x &= 2 \end{aligned}$$

Therefore, the parts are 7 and 2.

Ex. 2. A has £700, B has £600. How much must A receive from B in order that A's money may be double B's.

Suppose A receives £x.

Then A will have £$700+x$ (he had £700 to begin with).

B will have £$600-x$.

Now A's money is to be double B's.

$$\begin{aligned} \therefore \quad 700 + x &= 2(600 - x) \\ 700 + x &= 1200 - 2x \\ 3x &= 500 \\ x &= £166\tfrac{2}{3} \end{aligned}$$

Therefore, A must receive £166 13s. 4d.

EXERCISE L

(1) Divide £37 between A, B, and C, so that B has £3 more than A, and C £5 less than B.

(2) Find a number which, when multiplied by 4, becomes as much above 30 as it now is below it.

(3) Find a number which, when multiplied by 7, exceeds three times its original value by 20.

(4) Two men, at a distance of 24 miles, set out to meet each other, travelling at the rates of 3 and 5 miles an hour. When and where will they meet?

(5) A has £50 and B has £20. How much must A give B that they may have equal amounts?

(6) A has £50 and B has £40. How much must A take from B in order that A's share may be double B's?

(7) Find two consecutive numbers such that their sum is 91.

(8) A salmon weighed 12 lbs. and half its own weight. What was its weight?

(9) Find two numbers such that their sum is 40 and difference 16.

(10) Find two numbers whose sum is 34 and difference 14.

(11) Find two consecutive numbers such that the difference between 9 times the greater and 7 times the less is 29.

(12) Five times a certain number exceeds 4 times the next higher number by 8. Find the number.

(13) Divide £1025 between A, B, and C, so that A has £125 more than B, and B £300 more than C.

(14) A basket of apples is divided between 12 boys and 4 girls. If a boy has 2 more than a girl, and the total number of apples is 152, find how many each girl receives.

(15) A medicine bottle full of medicine costs $2\frac{1}{2}$d. If the medicine cost 2d. more than the bottle, what was the price of the bottle?

(16) A man promised to reap a field for 6s. an acre, but left off when there were still 4 acres unfinished, and received £2 14s. How many acres were there in the field?

(17) A boy has to write 50 lines each time he is late. In one term he had 430 lines to do, 80 of which were for coming without his books. How many times was he late?

(18) A party of boys hire a wagonette, and when it has to be paid for each boy's share is 10d. If there had been 4 more boys each one's share would have been 8d. How many boys were there, and how much was the hire of the wagonette?

(19) A party settling their bill for tea pay 8d. each. If there had been 4 more they would have paid 7d. each. How many people were there?

(20) Divide £1000 between B, C, and A, so that A has £72 more than B, and C £100 more than A.

(21) Divide £600 between three people, so that the first may have 4 times as much as the second, and the second twice as much as the third.

(22) Divide £120 between four people, so that the first has twice as much as the second, the second twice as much as the third, and the fourth as much as the other three together.

(23) Find two numbers such that if the former be increased by 16 it becomes double the latter, and the sum of the numbers is 83.

(24) A footman was hired for £20 a year and his livery. He was discharged at the end of eight months, and was only given £10 and his livery. What was his livery worth?

(25) Find the product and the sum of the squares of two numbers whose sum is 59 and difference 17.

(26) Find two numbers, such that their sum is 40, and seven times the greater is equal to 21 times the less.

(27) A man has two droves of pigs, each containing the same number. He sells 21 of one drove and 61 of the other, and then finds that he has twice as many of the first left as he has of the others. How many pigs had he originally in each drove?

(28) A bill of £57 was paid in sovereigns and half-crowns, and the number of coins was 106. How many were there of each?

(29) A's age is three times B's. B is twice as old as C. The sum of their ages is 54. How old were they?

(30) A boy, after spending 6d. more than half his pocket money, had 1d. more than one-third of it left. How much pocket money had he?

(31) The distance from Bedford to London is 49 miles. A man walks for 4 hours, and then doubles his rate and cycles the rest of the way in 5 hours. How fast did he walk?

(32) A man wished to give 5d. each to some waits, but found that he had 7d. too little money in his pocket, so gave them 4d. each, and had 10d. remaining. How many were there?

(33) Two brothers have the same income. One saves a quarter of his income, the other spends £140 a year more than his brother, and at the end of four years is £300 in debt. What is his income?

(34) At an election 370 people voted, and the candidate elected had a majority of 16. How many votes did he poll?

·93. Many problems involve the use of fractions. These are best solved by multiplying both sides of the equation obtained by the L.C.M. of the denominators of the fractions involved, and then proceeding as before.

Ex. 1. Find a number such that the sum of its third, fifth, and ninth parts is 58.

Let x be the required number.

Then its third, fifth, and ninth parts respectively are:—

$$\frac{x}{3},\quad \frac{x}{5},\quad \text{and}\quad \frac{x}{9}$$

$$\therefore\quad \frac{x}{3} + \frac{x}{5} + \frac{x}{9} = 58.$$

Multiplying both sides by 45, the L.C.M. of 3, 5, 9, we have

$$15x + 9x + 5x = 58 \times 45$$
$$29x = 58 \times 45$$

$$x = \frac{58 \times 45}{29} = 2 \times 45 = 90$$

Therefore, the required number is 90.

Ex. 2. A man bought a number of tops at 9 for 6d., and half as many more at 6 for 7d. He gave 2 away, and sold the rest at 8 for 11d., thereby gaining seven shillings and a penny. How many tops were there?

Let x be the number of tops at 9 for 6d.

Since 9 tops cost 6d., each costs

$$\frac{6}{9}\text{d.} \quad \text{or} \quad \frac{2}{3}\text{d.}$$

$$\therefore \qquad x \text{ cost } \frac{2x}{3}\text{d.}$$

He bought half as many at 6 for 7d.

$$\therefore \qquad \text{he bought } \frac{x}{2} \text{ at } \frac{7}{6}\text{d. each.}$$

and they cost $\quad \dfrac{x}{2} \times \dfrac{7}{6} = \dfrac{7x}{12}\text{d.}$

$$\therefore \qquad \text{total cost was } \frac{2x}{3} + \frac{7x}{12}\text{d.}$$

$$= \frac{15x}{12}\text{d.}$$

The total number was $x+\frac{x}{2}$, he gave away 2.

$$\therefore \qquad \text{Sold } x+\frac{x}{2}-2 \text{ at } \frac{11}{8}\text{d. each, producing}$$

$$\left(\frac{3x}{2}-2\right)\frac{11}{8}\text{d.}$$

But since the selling price exceeded the cost price by 85 pence (all amounts referred to in the problem being expressed in pence),

$$\left(\frac{3x}{2}-2\right)\frac{11}{8} - \frac{15x}{12} = 85$$

$$\frac{33x}{16} - \frac{11}{4} - \frac{15x}{12} = 85.$$

Multiplying up by 48, the L.C.M. of the denominators,

$$99x - 132 - 60x = 85 + 48 = 4080$$
$$39x = 4212$$
$$x = 108$$

Now, the whole number bought was

$$x + \frac{x}{2} = 108 + 54$$

$\therefore$ Number was 162.

Ex. 3. At what time between the hours of 5 and 6 are the hands of a clock together.

Suppose they are together at x minutes past 5.

The minute hand travels over 60 minute spaces, while the hour hand travels over 5. That is, it travels 12 times as fast as the hour hand. Therefore, while the minute hand travels x spaces the hour hand travels $\frac{x}{12}$. At 5 o'clock the minute hand is 25 spaces behind the hour hand, and at x minutes past they are together.

$$\therefore \qquad x = \frac{x}{12} + 25$$
$$12x = x + 300$$
$$11x = 300$$
$$x = 27\tfrac{3}{11}$$

Therefore, they are together at $27\frac{3}{11}$ minutes past 5.

EXERCISE LI

(1) Find a number such that its fourth part exceeds its tenth part by 27.

(2) Find a number such that the sum of its third and seventh parts is 40.

(3) What is the number whose third and tenth parts together are as much less than 15 as the fifth part is greater than 4?

(4) I spent 20 guineas on cloth, buying 10 yards, some at £2 a yard, and the rest at 30s. a yard. How much of each did I buy?

(5) At what times between the hours of 5 and 6 o'clock are the hands of a clock at right angles?

(6) At what time between 4 and 5 o'clock are the hands of a watch in a straight line?

(7) The hands of a clock are together at 12 o'clock. When will they next be together?

(8) The hands of a clock are in a straight line at 6 o'clock. When will this occur next?

(9) Two men, whose distance apart is 49 miles, set out to walk towards each other. The one walks at the rate of $3\frac{1}{4}$ miles an hour, the other at $3\frac{3}{4}$ miles an hour. When and where will they meet?

(10) A man spends £120 more than $\frac{2}{5}$ of his income and then has £70 less than half of it left. What is his income?

(11) A bill of £5 is paid with half sovereigns and half-crowns, the number of coins being 16. How many of each were used?

(12) A church collection amounted to £3 19s. 6d. The coins included were half-crowns, shillings, sixpences, and pence. There were four times as many shillings as half-crowns, and the number of sixpences was one less than four times the number of shillings and half-crowns together, and there were 786 coins in all. Find the number of each.

(13) I buy a number of eggs at 4 for 3d., five times as many at 1s. 2d. a score, and three times as many at 9d. per dozen. If I sell the lot at 5 for 4d., I gain 1s. 2d. How many of each kind did I buy?

(14) A bankrupt owes A three times as much as he owes B, and C as much as he owes the other two together. If he has £400 to divide between them, what should each receive?

(15) Divide £1500 between A, B, and C so that A's share is $\frac{3}{5}$ths of B's, and B's $\frac{5}{8}$ths of C's.

(16) How much chicory at 4d. per lb. must be mixed with 60 lbs. of coffee worth 1s. 6d. per lb. so that the mixture may be worth 11d. per lb.?

(17) A brewer had a quantity of beer which he expected to sell for £9 2s. He sold 40 gallons, but had to throw away a quarter of the remainder because it contained arsenic, and consequently he only received £7 7s. 4d. How many gallons were there, and how much a gallon did it fetch?

(18) The ages of a man and his wife together amount to 60 years, and 20 years ago the man's age was half as great again as his wife's. What are their present ages?

(19) The denominator of a certain fraction exceeds the numerator by 5, and if 1 be subtracted from both numerator and denominator the fraction becomes equal to $\frac{1}{2}$. What is the fraction?

(20) A boy has $2\frac{1}{4}$ hours for nutting. He rides to a wood on a bicycle at 10 miles an hour, spends $\frac{3}{4}$-hour getting nuts, and then finding his tyre punctured has to walk home at the rate of $3\frac{1}{3}$ miles an hour, just arriving in time. How far away is the wood?

CHAPTER XI

SIMULTANEOUS EQUATIONS

94. THE equations hitherto considered have contained only one unknown quantity, and the methods given will in such cases always effect a solution.

If, however, we are given an equation which, although of the first degree, contains more than one unknown quantity, we shall not be able to find a definite solution, or rather we shall find such a number of solutions or series of values for the unknown quantities that we shall not know which to take. Thus $5x-3y=1$ is satisfied by

$$\begin{aligned} x&=2, y=3\\ x&=5, y=8\\ x&=8, y=13 \end{aligned}$$

95. If there are two unknown quantities and two equations of the first degree containing them, this indefinite number of solutions disappears, and it is in general possible to find one and only one solution for each unknown quantity, which will satisfy both equations simultaneously. Equations of this description are known as *simultaneous equations*, and the method in use for their solution will best be understood from an example:—

Ex. 1.
$$\left.\begin{aligned}2x+3y&=19\\5x-7y&=4\end{aligned}\right\}\begin{aligned}&\text{(i.)}\\&\text{(ii.)}\end{aligned}$$

If we can get rid of the terms containing y from both equations we shall be left with a simple equation to find x, which can be easily solved.

It is clear that if the first equation contained the same number of y's as the second, addition would wipe out or eliminate the terms containing y. Hence, we make it our aim to do this, and by multiplying the first equation by the coefficient of y in the second, and the second by the coefficient of y in the first, this object will be attained thus :—

$$\left.\begin{aligned}7(2x+3y)&=19\times 7\\3(5x-7y)&=4\times 3\end{aligned}\right\}$$

$$\left.\begin{aligned}14x+21y&=133\\15x-21y&=12\end{aligned}\right\}$$

$$\text{adding}\quad 29x=145$$

$$x=\frac{145}{29}=5$$

But $\quad 2x+3y=19\quad$ by equation (i.),

and x has been found to be 5. Substituting this value for x,

$$\begin{aligned}\therefore\quad 2\times 5+3y&=19\\10+3y&=19\\3y&=19-10=9\\y&=3\end{aligned}$$

96. Sometimes it may be found more advantageous to eliminate x, but the method is precisely similar.

Ex. 2.
$$\left.\begin{aligned}7x+10y&=41\\13x-11y&=17\end{aligned}\right\}\begin{aligned}&\text{(i.)}\\&\text{(ii.)}\end{aligned}$$

Multiplying (i.) by 13, and (ii.) by 7, we have

$$\begin{aligned}91x+130y &= 533\\ 91x-\ \ 77y &= 119\\ \hline 207y &= 414\\ y &= \frac{414}{207} = 2\end{aligned}$$

But from (i.) $7x+10y=41$

∴ substituting for y $7x+10\times 2=41$

$$\begin{aligned}7x+20&=41\\ 7x&=41-20=21\\ x&=3\end{aligned}$$

EXERCISE LII

(1) $\left.\begin{aligned}x+y&=9\\3x+5y&=35\end{aligned}\right\}$ (2) $\left.\begin{aligned}2x+3y&=18\\3x-2y&=1\end{aligned}\right\}$

(3) $\left.\begin{aligned}2x-y&=8\\x-2y&=1\end{aligned}\right\}$ (4) $\left.\begin{aligned}2x-9y&=11\\3x-12y&=15\end{aligned}\right\}$

(5) $\left.\begin{aligned}5x+6y&=34\\3x-y&=2\end{aligned}\right\}$ (6) $\left.\begin{aligned}3x-7y&=7\\11x+5y&=87\end{aligned}\right\}$

(7) $\left.\begin{aligned}14x-3y&=1\\11x+2y&=40\end{aligned}\right\}$ (8) $\left.\begin{aligned}9x-4y&=8\\13x+7y&=101\end{aligned}\right\}$

(9) $\left.\begin{aligned}2x-y&=3\\2x+3y&=5\end{aligned}\right\}$ (10) $\left.\begin{aligned}2x+3y&=13\\5x+4y&=22\end{aligned}\right\}$

(11) $\left.\begin{aligned}13x+11y&=81\\2x+y&=9\end{aligned}\right\}$ (12) $\left.\begin{aligned}12x+32y&=340\\4x+12y&=124\end{aligned}\right\}$

(13) $\left.\begin{aligned}5x+7y&=43\\11x+9y&=69\end{aligned}\right\}$ (14) $\left.\begin{aligned}8x-21y&=33\\6x+35y&=177\end{aligned}\right\}$

(15) $\left.\begin{aligned}19x-18y&=23\\18x-19y&=14\end{aligned}\right\}$ (16) $\left.\begin{aligned}10y-12x&=6\\5x+6y&=28\end{aligned}\right\}$

(17) $\left.\begin{aligned}13y-100&=6x\\13x-55&=y\end{aligned}\right\}$ (18) $\left.\begin{aligned}41x-40y&=49\\17x-13y&=49\end{aligned}\right\}$

(19) $\left.\begin{aligned}20y&=10x\\17x-32y&=12\end{aligned}\right\}$ (20) $\left.\begin{aligned}x-11y&=-2\\y-11x&=-218\end{aligned}\right\}$

In the following do not clear of fractions, but treat $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities, and find them both by elimination and substitution, afterwards giving the values of x and y.

(21) $$\frac{1}{x} + \frac{1}{y} = 9, \quad \frac{1}{x} - \frac{1}{y} = 1$$

(22) $$\frac{3}{x} + \frac{2}{y} = 2, \quad \frac{2}{x} + \frac{1}{y} = \frac{7}{6}$$

(23) $$\frac{10}{x} + \frac{12}{y} = 8, \quad \frac{5}{x} - \frac{2}{y} = 0$$

(24) $$\frac{3}{x} + \frac{2}{y} = 2, \quad \frac{2}{x} - \frac{3}{y} = -\frac{5}{6}$$

96. In most cases, however, where fractions occur in the equations, it is best to clear of fractions by multiplying both sides of each equation by the L.C.M. of the denominators of the fractions occurring therein, and then, after collecting like terms, proceeding as before.

Ex. $$\left.\begin{aligned} \frac{x+2}{7} + \frac{y-x}{4} &= 2x - 8 \\ \frac{2y-3x}{3} + 2y &= 3x + 4 \end{aligned}\right\} \begin{aligned} &\text{(i.)} \\ &\text{(ii.)} \end{aligned}$$

Multiplying (i.) by 28, the L.C.M. of 7 and 4, and (ii.) by 3, we have

$$\left.\begin{aligned} 4(x+2)+7(y-x) &= 28(2x-8) \\ 2y-3x+6y &= 9x+12 \end{aligned}\right\}$$

i.e. $$\left.\begin{aligned} 4x+8+7y-7x &= 56x-224 \\ 2y-3x+6y &= 9x+12 \end{aligned}\right\}$$

or collecting like terms,

$$\left.\begin{aligned} -59x+7y &= -\ 232 \\ -12x+8y &= 12 \end{aligned}\right\} \begin{matrix}\text{(i.)}\\ \text{(ii.)}\end{matrix}$$

which may be solved as before by multiplying (i.) by 8 and (ii.) by 7, and subtracting.

EXERCISE LIII

(1) $$\left.\begin{aligned} 8x-3y &= 19+\frac{7x+1}{8} \\ 3x-5y &= 18-\frac{13y-3}{11} \end{aligned}\right\}$$

(2) $$\left.\begin{aligned} x-\frac{y-2}{7} &= 5 \\ 4y-\frac{x+10}{3} &= 3 \end{aligned}\right\}$$

(3) $$\left.\begin{aligned} \frac{4x+3y}{6} &= 8 \\ \frac{7y-3x}{2}-y &= 11 \end{aligned}\right\}$$

(4) $$\left.\begin{aligned} 2x-\frac{y-3}{5} &= 4 \\ 3y+\frac{x-2}{3} &= 9 \end{aligned}\right\}$$

(5) $$\left.\begin{aligned} \frac{2x-y}{7}+3x &= 2y-6 \\ \frac{y+3}{5}+\frac{y-x}{6} &= 2x-8 \end{aligned}\right\}$$

(6) $$\left.\begin{aligned} \frac{4x+5y}{40} &= x-y \\ \frac{2x-y}{3}+2y &= \frac{1}{2} \end{aligned}\right\}$$

(7) $$\left.\begin{aligned} 2y - \frac{x-6y+1}{7} &= \frac{x-3}{2} \\ \frac{x-5y+8}{9} &= \frac{3x-13y}{7} \end{aligned}\right\}$$

(8) $$\left.\begin{aligned} \frac{x-y}{5} &= 2 \\ \frac{3x-7y}{3} &= \frac{2x+y+1}{5} \end{aligned}\right\}$$

(9) $$\left.\begin{aligned} \frac{2x}{3} - 4 + \frac{y}{2} + x &= 8 - \frac{3y}{4} + \frac{1}{12} \\ \frac{y}{6} - \frac{x}{2} + 2 &= \frac{1}{6} - 2x + 6 \end{aligned}\right\}$$

(10) $$\left.\begin{aligned} \frac{1}{2}\left(x-y\right) &= \frac{1}{4}\left(x+y\right) \\ \frac{3}{5}\left(3x-4y\right) &= 3 \end{aligned}\right\}$$

(11) $$\left.\begin{aligned} \frac{x-2}{5} - \frac{10-x}{3} &= \frac{y-10}{4} \\ \frac{2y+4}{3} - \frac{2x+y}{8} &= \frac{x+13}{4} \end{aligned}\right\}$$

(12) $$\left.\begin{aligned} \frac{5x-6y}{13} + 3x &= 4y - 2 \\ \frac{5x+6y}{6} - \frac{3x-2y}{4} &= 2y - 2 \end{aligned}\right\}$$

CHAPTER XII

SIMULTANEOUS EQUATIONS WITH MORE THAN TWO UNKNOWNS

97. It was pointed out in Chapter XI. that the equation $5x-3y=1$ was satisfied by a number of different pairs of values (2, 3), (5, 8), (8, 13) of x and y. The student will easily find by trial that the same thing holds for any simple equation with two unknowns, for he may assign any value he likes to y, and ascertain the corresponding value for x. When, however, a second equation between x and y was coupled with the first one, we found that in general only one pair of values could be found that satisfied the two equations simultaneously. These considerations may be extended to the case of *two* equations between *three* unknowns. Thus, if we are given the two equations

$$\left.\begin{aligned}2x+3y+5z&=24\\5x-7y-3z&=\ \ 1\end{aligned}\right\}$$

any number of sets of values can be found which satisfy them simultaneously, for we may give any value we like to z, and then find out what the corresponding values of x and y are. If, for instance, we put $z=1$, the equations reduce to

$$\begin{aligned}2x+3y&=19\\5x-7y&=\ \ 4\end{aligned}$$

which have already been solved in section 95, when it is shown that $x=5$, $y=3$. The student may satisfy himself by actual trial that the sets of values (31, 34, − 28), (− 21, − 28, 30), as well as the set (5, 3, 1) also satisfy both equations.

It will be found, however, that if we couple a third equation $x+y+z=9$ with the two already given, the set of values (5, 3, 1) is the only one which satisfies the three simultaneously.

98. The solution of the set of three

$$\left.\begin{aligned} 2x+3y+5z&=24 \\ 5x-7y-3z&=1 \\ x+y+z&=9 \end{aligned}\right\}\begin{aligned}&\text{(i.)}\\&\text{(ii.)}\\&\text{(iii.)}\end{aligned}$$

is effected by the same methods as are employed in Chapter XI.

Eliminate z between (i.) and (iii.)

$$\begin{aligned} 5x+5y+5z&=45 \\ 2x+3y+5z&=24 \\ 3x+2y&=21 \text{ (iv.)} \end{aligned}$$

Eliminate *the same letter* z between (ii.) and (iii.)

$$\left.\begin{aligned} 5x-7y-3z&=3 \\ 3x+3y+3z&=27 \end{aligned}\right\}$$

$$\begin{aligned} 8x-4y&=30 \\ 4x-2y&=15 \text{ (v.)} \end{aligned}$$

Now solve (iv.) and (v.) simultaneously for x and y, and we obtain $x=5$, $y=3$.

Substituting these values in (iii.)

$$\begin{aligned} 5+3+z&=9 \\ z&=1 \end{aligned}$$

It must be carefully noted that we reduce *three* equations containing *three unknowns* to *two equations* containing *the same two unknowns* (in this case x and y). It may happen that one of the given equations contains only two of the three unknowns. In that case we eliminate the *third unknown* between the other two. Thus, in the set

$$\left.\begin{aligned} 2x+3y+4z&=20\\ 3x+4y+2z&=17\\ 5x+3z&=14\end{aligned}\right\}\begin{matrix}\text{(i.)}\\ \text{(ii.)}\\ \text{(iii.)}\end{matrix}$$

we notice that y is wanting in (iii.). Hence, we at once proceed to eliminate y between (i.) and (ii.), obtaining $-x+10z=29$ (iv.). Solving (iii.) and (iv.) simultaneously, we get $x=1$, $z=3$, and then by substitution in (i.) or (ii.) $y=2$.

Similarly, if we are given *four equations* between *four unknowns* we reduce to *three equations* between *three unknowns*.

EXERCISE LIV

(1) $\left.\begin{aligned} x+3y+z&=11\\ 2x+y+z&=8\\ 3x+2y+z&=11\end{aligned}\right\}$

(2) $\left.\begin{aligned} x+2y+4z&=28\\ 2x+y+3z&=22\\ x+2y+2z&=18\end{aligned}\right\}$

(3) $\left.\begin{aligned} x-y+z&=4\\ 2x+3y-2z&=8\\ 3x-2y+3z&=16\end{aligned}\right\}$

(4) $\left.\begin{aligned} 3x+4y+5z&=12\\ 2x+7y-3z&=6\\ 5y+4z&=9\end{aligned}\right\}$

(5) $\left.\begin{aligned} x+y&=8\\ y+z&=6\\ x+z&=4\end{aligned}\right\}$

(6) $\left.\begin{aligned} \frac{6}{x}+\frac{10}{y}+\frac{12}{z}&=9\\ \frac{2}{x}+\frac{5}{y}+\frac{3}{z}&=3\\ \frac{4}{x}+\frac{20}{y}+\frac{9}{z}&=9\end{aligned}\right\}$

(7) $\left.\begin{aligned} x+y+z&=6\\ y+z+w&=7\\ z+w+x&=8\\ x+y+w&=9\end{aligned}\right\}$

(8) $\left.\begin{aligned} \frac{x+y}{7}+\frac{y+z}{6}&=5\\ \frac{y+z}{3}+\frac{z+x}{4}&=10\\ \frac{z+x}{2}+\frac{x+y}{2}&=15\end{aligned}\right\}$

99. Problems involving the use of two or more unknowns are attacked in the same manner as those involving one unknown only. The difficulty of the student will lie rather in *forming* his equations than *in solving them* when formed. As in the cases previously treated, the formation of the equations consists in translating into algebraical language what is given in English.

Thus, if we are asked *to find a fraction which becomes* $\frac{1}{5}$ *if we add unity to its numerator, but* $\frac{1}{7}$ *if we add unity to its denominator,* we translate into Algebra thus:—

$$\left.\begin{array}{l}\text{Let } x \text{ be the numerator}\\ \text{Let } y \text{ be the denominator}\end{array}\right\}\text{of the required fraction,}$$

then
$$\left.\begin{aligned}\frac{x+1}{y}&=\frac{1}{5}\\ \frac{x}{y+1}&=\frac{1}{7}\end{aligned}\right\}$$

Clearing of fractions as on p. 122, we find $x=3$, $y=20$, and hence that the fraction is $\frac{3}{20}$.

EXERCISE LV

1. Find two numbers such that half the first with a third part of the second may make 16, and that a quarter of the first with a fifth part of the second may make 9.

2. If A gave B 5 shillings, B would have three times as much as A had left; while, if B gave A 5 shillings, A would have twice as much as B had left. What had each?

3. Find the area of a rectangular field such that if its length and breadth were each increased by one chain, the area would be increased by 14 square chains; but

if its length were increased and its breadth diminished by one chain, its area would be diminished by four square chains.

4. Divide the number 22 into three such parts that we get the same number when we add 5 to the first, when we subtract 5 from the second, and when we multiply the third by 5.

5. A cistern can be emptied by two taps open together in 2 hours. But if one were closed when they had emptied half of it, it would take the other one 3 hours longer to empty the cistern. Find how long each would take by itself.

6. A certain fraction becomes $\frac{1}{3}$ if 3 be added to the numerator, but becomes $\frac{1}{6}$ if 3 be added to the denominator. Find it.

7. A man can scull from P to Q, a distance of 15 miles, and back again in 8 hours. Find the rate at which he can scull in still water, and the rate at which the stream is flowing, if he is 2 hours longer going up than coming down.

8. If 10 oxen and 15 sheep cost £100, and 13 oxen and 4 sheep cost £99, find the price of an ox and a sheep.

EXERCISES FOR REVISION

The letters "G.C.M." sometimes occur in Examination Papers where "H.C.F." might be expected. See Art. 70. We have in some cases allowed them to remain as given.

I.

(1) Find the value of

(i.) $a^2+b^2+c^2+2ab+2ac+2bc$
(ii.) $a^3+b^3+c^3-3abc$
(iii.) $a+ab+abc+abcd$
when $a=1,\ b=2,\ c=-3,\ d=0$

(2) Multiply

(i.) $5x^2-7xy+3y^2$ by $2x^2+4xy-y^2$
(ii.) $a^2+3ab-4c^2$ by $a^2-3ab+4c^2$

(3) Divide

(i.) $x^5+7x^4+20x^3+26x^2+12x$ by x^2+3x+2
and (ii.) $a^3+8b^3-c^3+6abc$ by $a+2b-c$

(4) Simplify

(i.) $2[3a-4b-\{2a-3b-2(a-b)\}]$
(ii.) $3(x-1)(x-2)-2(x-3)(x-4)$

(5) Solve the equations

(i.) $2(x+1)+3(x+2)=23$
(ii.) $5(x-7)-3(x+4)=-33$

II.

(1) Find the value of

(i.) $a^2+b^2+c^2-2ab-2ac+2bc$
(ii.) $a^3+b^3-c^3+3abc$
(iii.) $a-ab+abc-abcd$
when $a=-2,\ b=2,\ c=0,\ d=-1$

(2) Multiply

(i.) $3x^2 - 2xy + 4y^2$ by $5x^2 + 3xy - 1y^2$
(ii.) $a^2 - 3ab - 4c^2$ by $a^2 + 3ab - 4c^2$

(3) Divide

(i.) $x^5 - 7x^4 + 20x^3 - 26x^2 + 12x$ by $x^2 - 3x + 2$
and (ii.) $a^3 - 8b^3 + c^3 + 6abc$ by $a - 2b + c$

(4) Simplify

(i.) $2[3a - 4b + \{2a + 3b - 2(a - b)\}]$
(ii.) $4(x - 1)(x + 2) - (x - 3)(x + 4)$

(5) Solve the equations

(i.) $5(x + 1) + 2(x + 2) = 30$
(ii.) $6(x - 7) - 2(x - 4) = 6$

III.

(1) Find the value of each of the following expressions when $a = 1,\ b = -2,\ c = 0,\ d = 3$

(i.) $(a - b) + (b - c) + (c - d) + (d - a)$
(ii.) $(a - b)^2 + (b - c)^2 + (c - d)^2 + (d - a)^2$
(iii.) $(a - b)^3 + (b - c)^3 + (c - d)^3 + (d - a)^3$
(iv.) $(a - b)\ (b - c)\ (c - d)\ (d - a)$

(2) Explain carefully the difference between

(i.) $a + bc$ and $(a + b)c$ (ii.) $2x^2$ and $(2x)^2$
(iii.) $(a + b)^2$ and $a + b^2$ (iv.) x^3 and $3x$

(3) Multiply $2bc + b^2 + c^2 - a^2$ by $2bc - b^2 - c^2 + a^2$

(4) Divide

(i.) $a^3 - 3a^2b + 3ab^2 - b^3 + c^3$ by $a - b + c$
and (ii.) $(a - b)^2 - (x - y)^2$ by $a - b + x - y$

(5) Simplify $(a - x)^2 + 2(a - x)(x - b) + (x - b)^2$

(6) Solve the equations

(i.) $5(x - 3) + 7(x - 11) = 4(x + 1)$
(ii.) $5(x - 3) - 7(x - 11) = 38$

(7) Show that

(i.) $(x-y)^2+(x-y)(y-z)+(y-z)^2$
(ii.) $(y-z)^2+(y-z)(z-x)+(z-x)^2$
and (iii.) $(z-x)^2+(z-x)(x-y)+(x-y)^2$

and all equal to one another.

IV.

(1) Find the quotient and remainder when
x^4+ax^3+bx+1 is divided by x^2+x+1

(2) Find the factors of

(i.) x^2+x-72 (ii.) $x^2-3bx-40b^2$
(iii.) $3x^3-27x$ (iv.) $ab+ac+bd+cd$
(v.) $x^4+15x^3+56x^2$

(3) Solve the equation

$$11(7-2x)-13(5+x)+17(3x+4)=0$$

(4) Find the G.C.M. of
$x^4-3x^2-14x-12$ and $x^4-4x^3-8x+32$

And the L.C.M. of
$x^4+5x^3+6x^2$, $x^4-5x^3+6x^2$, and x^4-13x^2+36

(5) Two numbers together make 29, and twenty-five times the smaller is equal to four times the greater. Find the numbers.

(6) If $y+xy^3=18$ and $y=2$, find the value of x

(7) Find the value when $x=-\frac{169}{3}$, $y=-\frac{9}{13}$ of

(i.) $xy+xy^2$ (ii.) $y+xy^3$

V.

(1) Find the value of

$$\frac{3a+4b}{3x-a-b}-\frac{a^2-2b^2}{x^2-b^2} \text{ when } a=10,\ b=5,\ x=7$$

(2) Multiply $2a+3b-4c$ by $2a-3b-4c$

(3) Divide $3x^4+14x^3+9x+2$ by x^2+5x+1

(4) Find the highest common factor of

$$x^2+11x+30 \text{ and } x^3+8x^2+9x-18$$

(5) Simplify

$$3(x+z)-(6y-z)-2\{x-(2y+z)-(y-3z)\}$$

(6) Solve the equations

(i.) $3(x-7)-5(x-9)=4$
(ii.) $(x-2)(x+3)+(x-7)(x+7)=2(x-2)^2$

(7) Show that the sum of the squares of two quantities is greater than the square of their difference by twice their product.

VI.

(1) The difference between two expressions is $3a+2b-c$. If the smaller is $5a-b+x$, what is the larger?

(2) Multiply

(i.) $3a+2b$ by $2a-3b$
(ii.) $3a^2+a-2$ by $2a^2-3a-4$

(3) What is the value of $abc^2-3a^2b+7a^3c$ when

$$a=3,\ b=-2, \text{ and } c=1?$$

(4) The quotient of two expressions is $3x^2-2x+1$. The divisor is $3x-7$. What is the other expression?

(5) Simplify

$$2a-3\{a-4[a-2(b+c)-b]+c\}+3[a-2(b-c)]$$

(6) Solve $3(x-5)=2(x-8)-7(x-3)$

VII.

(1) Add together

(i.) $2a-3b+4c$, $-5a+2b-c$, $7a-3b-7c$
(ii.) x^2y+xy^2+3x, $5x-xy^2+4x^2y$, $7xy^2-10x^2y+9x$

(2) Find the value when $x = 2$, $y = 3$, $a = -1$ of
(i.) $2x + 3y + 4a$ (ii.) $ax + 2xy - 3ay + 15$

(3) Subtract
twice $5x - 7y + 13z$ from three times $11x - 8z$

(4) Multiply
(i.) $2x - 3$ by $7 - 5x$
(ii.) $x^2 - 5xy + 3y^2$ by $2x^2 + xy - 4y^2$
(iii.) $2x - 3a$ by $3x - 2a$
(iv.) $x^2 - 3xy + y^2$ by $2x^2 + xy - 3y^2$

(5) Simplify
(i.) $5(a - b) - 6(2a + b) + 7(b - 3a)$
(ii.) $-[-(a - x)] + [-(a + x)] - [-(-a + x)]$

(6) Divide
(i.) $x^4 - x^2 + 2x - 1$ by $x^2 + x - 1$
(ii.) $x^4 - 81y^4$ by $x - 3y$

(7) If the value of a placed goal be x points, of a dropped goal y points, and of a try be z points, what is the algebraical expression for the score of a winning team which gained three placed goals, one dropped goal, and four tries? If their opponents scored one placed goal, two dropped goals, and five tries, by how many points did they lose?

VIII.

(1) If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, find the values of

(i.) $\dfrac{12e - cd}{12} + \dfrac{4c^2}{a^3 + b^3} - \dfrac{b^2 + c^2 + cd}{5a^2}$

(ii.) $(a^2 - c^2 - d)\sqrt{d^3}$

(2) Simplify
(i.) $2a^3 - (b^3 - c^3) - [a^3 + \{ b^3 - (c^3 + b^3) \}] - \overline{a^3 - b^3}$
(ii.) $7\{ a^2 - 5(a^2 - 2b) \} + 30a^2$

(3) Multiply $x^2 + 2ax + 3a^2$ by $x^2 - 2ax + a^2$
and divide $a^4 + 4b^4$ by $a^2 + 2ab + 2b^2$

(4) Find the G.C.M. of

$x^4 - 2x^2 + 1$ and $x^4 - 4x^3 + 6x^2 - 4x + 1$

(5) Find the L.C.M. of $6x^2 + 13x + 6$ and $8x^2 + 6x - 9$

(6) Solve $13x - 21(x - 3) = 10 - 21(3 - x)$

(7) A number has two digits: if 9 be added to it, the digits are reversed; if the number be divided by the sum of the digits, the quotient is equal to 4. Find the number.

IX.

(1) Express the sum of a pounds and b ounces (avoirdupois) in drams.

A man buys p ducks for q shillings. How much does each cost him?

(2) Find the value of $\dfrac{a^4 + a^2b^2 + b^4}{a^2 - ab + b^2}$ when $a = 3$, $b = 2$

(3) Simplify

(i.) $(x + 3)^2 + 2(x + 3) + 1$
(ii.) $(x - 3)^2 + 2(x - 3) + 1$
(iii.) $(x + 3)^2 - 2(x + 3) + 1$
(iv.) $(x - 3)^2 - 2(x - 3) + 1$

(4) Multiply

(i.) $x^2 - 6x + 9$ by $x^2 + 6x + 9$
and (ii.) $x^4 + 4ax^3 + 6a^2x^2 - 4a^3x + a^4$ by $x - a$

(5) Divide

(i.) $x^5 - 5ax^4 + 10a^2x^3 - 10a^3x^2 + 5a^4x - x^5$
by $x^3 - 3ax^2 + 3a^2x - a^3$
and (ii.) $x^8 - 18x^4 + 81$ by $x^4 - 6x^2 + 9$

(6) Solve the equations

(i.) $2x - 7 = x + 11$ (ii.) $5(x - 3) = 3(x - 1)$
(iii.) $5(x - 3) - 3(x - 5) = 0$

(7) Find two numbers whose sum is 12, and whose difference is equal to the smaller number.

X.

(1) Add together

(i.) $5a-3b+4c, \quad -5a+4b-c, \quad 7a-3b+3c$

(ii.) $x^2y+xy^2+4x, \quad 5x-3xy^2+4x^2y,$
$9xy^2-10x^2y+9x$

(2) Find the value when $x=2$, $y=3$, $a=-1$ of

(i.) $7x-4a-6y$ (ii.) $xy+ay-ax$

(3) Subtract four times $5x-7y+13z$ from five times $11x-8z$.

(4) Multiply

(i.) $3x-8$ by $9-6x$

(ii.) $2x^2-5xy+3y^2$ by $x^2+xy-4y^2$

(iii.) $x^2-3xy+y^2$ by $2x^2+xy-3y^2$

(5) Simplify

(i.) $4(a-2b)+6(2a-b)-7(b-8a)$

(ii.) $-[-(a+x)]+[-(a-x)]-[-(-a-x)]$

(6) Divide

(i.) $x^4-3x^3+6x^2-9x+9$ by x^2-3x+3

(ii.) a^4+4b^4 by $a^2+2ab+2b^2$

(7) A man gains in each year b pounds more than in the previous year. If in the first year he gains a pounds, find what he gains (i.) in the third year, (ii.) in 3 years.

XI.

(1) If $a=2$, $b=1$, $c=0$, $d=3$, find the value of

$$abc+abd-acd+2ab^2+d^3$$

(2) Add together

$$12a^3+5a^2+a-10, \quad a^3+4a-12, \quad 6a^2+7a, \text{ and } 2a^2+a+9$$

and subtract

$7x^3-3x^2y+5xy^2-6y^3$ from $9x^3-2x^2y+4y^3$

(3) Multiply

$$3a^3 - 2a^2x - 4x^3 \text{ by } 2a^2 - 7ax + x^2$$

and divide $2x^3 + 3x^2 - 2x$ by $x^2 - 1$

(4) Simplify

$$a + 2b + 3c + (2a - 4b + 7c) - (a + 2b - c)$$

(5) Find the G.C.M. of

(i.) $x^2 - 3x + 2$ and $x^2 - x - 2$
(ii.) $x^3 + 3x^2 + 4x + 12$ and $x^3 + 4x^2 + 4x + 3$

(6) Solve the equations

(i.) $40 - 2(3x + 8) = 1 + 7(17 - 2x)$
(ii.) $ax = ab - a$

(7) Divide £1800 into two parts, so that seven times one part shall be double the other.

XII.

(1) Define term, factor, coefficient, root, power.

(2) Find the value of

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \text{ when } x = \cdot 3 \quad y = \cdot 7$$

(3) Multiply $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^3 + 3x^2y + 3xy^2 + y^3$

(4) Simplify

(i.) $(x + 2y) + (3x + 5y) + (7x - 8y)$
(ii.) $(x + 2y) - (3x + 5y) + (7x - 8y)$
(iii.) $(x + 2y)(3x + 5y) - (3x + 5y)(7x - 8y)$
(iv.) $3(x + 2y) + 6(3x + 5y) - (7x - 8y)$
(v.) $(x + y)(x + 2y)(x + 3y) - x^3 - 6y^3$

(5) Divide

$$x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$$
$$\text{by } x^3 - 3x^2y + 3xy^2 - y^3$$

(6) Solve (i.) $5(x - 7) + 11(x - 8) = 5$
(ii.) $5(x - 19) - 13(x - 20) = 5$

(7) Divide £280 among A, B, and C, so that B shall have £20 more than C, and A shall have £30 more than B.

XIII.

(1) Simplify

$$3\{x-5[x-2(3-2\overline{x-1})+4]-x\}$$

(2) Solve

(i.) $5x-4=2x-13$
(ii.) $5x-3[x-8(x-7)+2]=8$

(3) Solve

(i.) $(3x-8)(2x+1)-5(x-2)^2=(x-2)(x+4)$
(ii.) $\dfrac{x+3}{5}-\dfrac{2x-5}{3}=\dfrac{x-9}{2}$

(4) Find the H.C.F. of

$$6x^3-7x^2-16x+12 \text{ and } 4x^3-8x^2-9x+18$$

(5) Find the L.C.M. of

$$x^3-8y^3,\ x^2-4y^2,\ x^2-xy-2y^2, \text{ and } x^2+3xy+2y^2$$

(6) Simplify

(i.) $\dfrac{1}{x-\dfrac{2}{x+\frac{1}{2}}}\times\dfrac{1}{2+\dfrac{1}{x}}\div\dfrac{x}{2x-\dfrac{x+4}{x+1}}$

(ii.) $\dfrac{2y}{x^2-y^2}-\dfrac{1}{x-y}+\dfrac{1}{x+y}+\dfrac{1}{x}-\dfrac{1}{y}$

(iii.) $\dfrac{a^3-(a-b)xy}{a^3-ab^2}+\dfrac{xy}{a(a+b)}-\dfrac{a}{2a-2b}$

(7) Write down an expression of the fourth degree which shall leave the same remainder whether it be divided by $x+1$, $x+2$, or $x+3$.

XIV.

(1) Find the value when $a=2$, $b=-3$, $c=4$, of

(i.) $(a-b)^3+(b-c)^3+(c-a)^3$
(ii.) $3(a-b)(b-c)(c-a)$
(iii.) $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$
(iv.) $a^3-3a^2(b+c)+3a(b+c)^2-(b+c)^3$

(2) Use the theorem $a^2+2ab+b^2=(a+b)^2$ to simplify the expression $(x-y)^2+2(x-y)(y-z)+(y-z)^2$

(3) Simplify

(i.) $\dfrac{a^2}{a+x} + x-a$

(ii.) $\dfrac{1}{3(1+x)} + \dfrac{2+x}{3(1-x+x^2)} - \dfrac{2}{1+x^3}$

(iii.) $\dfrac{1}{2(x-1)} - \dfrac{1}{x-2} + \dfrac{1}{2(x-3)}$

(iv.) $\dfrac{1}{(x-2)(x-3)} - \dfrac{2}{(x-1)(x-3)} + \dfrac{1}{(x-1)(x-2)}$

(v.) $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$

(4) Find the factors of

(i.) $49x^2+y^2-70x^3+25x^4-14xy+10x^2y$
(ii.) $4a^2b^2+(a^2-b^2)^2$

(5) Find the G.C.M. of

(i.) x^3-5x^2+7x-3 and x^2+x-12
(ii.) x^4-1 and x^3+4x^2-5
(iii.) $2a^3+3a^2x-9ax^2$ and $6a^3x-17a^2x^2+14ax^3-3x^4$

(6) Find the L.C.M. of

(i.) x^3+1 and $(x+1)^3$
(ii.) a^2-b^2, $(a-b)^2$, and a^3+b^3
(iii.) $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$ and x^5-ax^4
$+a^2x^3-a^3x^2+a^4x-a^5$

(7) Solve the equations

(i.) $\dfrac{6x+13}{15} - \dfrac{3x+5}{5x-25} = \dfrac{2x}{5}$

(ii.) $\dfrac{17-3x}{5} - \dfrac{2x+1}{1\frac{1}{2}} = \dfrac{29-11x}{3}$

XV.

(1) Given $3x=1$, $2y=-3$, $z=-4$, find the numerical values of the following expressions

(i.) $6xy-4yz+2zx$

(ii.) $\dfrac{9x}{8y^2} + \dfrac{24y}{z^2} + \dfrac{z}{36x^2}$

(iii.) $\dfrac{3x}{y+z} + \dfrac{z}{x+y}$

(2) Multiply

$$\frac{x^4}{16} + \frac{x^3}{4} - \frac{x}{2} - 1 \text{ by } \frac{x^2}{4} - \frac{x}{2} + 1$$

(3) Divide $(a+b)^3+c^3+d^3-3(a+b)cd$ by $a+b+c+d$

(4) Find the factors of the following expressions, and write down their least common multiple

(i.) $2x^2+3x-2$ (ii.) $(5x-7)^2-(x-5)^2$

(iii.) $2x^3-x^2-8x+4$

(5) Simplify the following expressions

(i.) $\dfrac{\dfrac{x}{x+1} + \dfrac{x}{x-1}}{\dfrac{2}{x^2-1}} - \dfrac{4x-\dfrac{1}{x}}{2+\dfrac{1}{x}}$

(ii.) $\left(\dfrac{1}{x^3-y^3} - \dfrac{1}{x^3+y^3}\right) \div \left(\dfrac{1}{x-y} - \dfrac{1}{x+y} - \dfrac{2y}{x^2+y^2}\right)$

(6) Solve the following equations

(i.) $\frac{x-1}{3} - \frac{2x-7}{4} = \frac{3}{4}$

and find the relation between a and b, when

(ii.) $\frac{3}{x-a} - \frac{2}{x+a} = \frac{x+b}{x^2-a^2}$

(7) A tradesman sends in a bill for £12, part of which is for labour, and the other part for materials; if the charge for labour had been twice what it was, and the charge for materials one-third of what it was, the amount of the bill would still have been £12. What was the charge for labour and what for materials?

XVI.

(1) Find the continued product of

(i.) $a^2 - ab + b^2$, $a^2 + ab + b^2$, and $a^4 + a^2b^2 + b^4$

(ii.) $x^2 + 2x + 3$, $x^2 - 2x + 3$, and $x^4 - 2x^2 + 9$

(2) If $x = 15$, what value must be given to y so as to make $5x - 7y = 19$. If we increased the value of x by 14, by how much should we have to increase the value of y so that the statement $5x - 7y = 19$ should still be true?

(3) There are two numbers such that if seven times the second be taken from five times the first the remainder is 38.

Express the above statement algebraically, and find any pair of numbers which have the property stated.

(4) Simplify

(i.) $\frac{2}{x-1} - \frac{8}{2(x-4)} + \frac{9}{x-3}$

(ii.) $\frac{1}{(a-b)(a-c)} - \frac{1}{(a-b)(b-c)} + \frac{1}{(a-c)(b-c)}$

(iii.) $\dfrac{x^4+x^2+1}{x^2+x+1} \times \dfrac{x^4+x^2+1}{x^2-x+1}$

(iv.) $\dfrac{x}{1-6x} + \dfrac{x}{1-2x} - \dfrac{2x}{1-3x}$

(v.) $\dfrac{a}{(a-b)(a-c)} - \dfrac{b}{(a-b)(b-c)} + \dfrac{c}{(a-c)(b-c)}$

(5) Express the sum of x pounds y ounces (i.) in oz., (ii.) in lbs.

What is the amount on p pounds for q years at r per cent. per annum?

(6) If the length of a right-angled parallelogram is $a+b-c$ inches and its breadth $a-b+c$ inches, find its area in square inches.

(7) If $a+b+c=0$, show that

$$(a^2+b^2+c^2)^2 = 4(b^2c^2+c^2a^2+a^2b^2)$$

XVII.

(1) Multiply together

$$1+ax,\ 1-ax,\ 1+\frac{x}{a},\ 1-\frac{x}{a}$$

and resolve $(x^2+a-1)^2-a^2x^2$ into its simple factors.

(2) Simplify the expressions

(i.) $\dfrac{(a+b)^5-a^5-b^5}{(a+b)^3-a^3-b^3}$

(ii.) $\dfrac{1+2x-3x^2}{1-3x-2x^2+4x^3}$

(iii.) $\dfrac{(ax+by)^2+(ay-bx)^2}{\left(\dfrac{a}{x}+\dfrac{b}{y}\right)^2+\left(\dfrac{a}{y}-\dfrac{b}{x}\right)^2}$

(iv.) $\dfrac{\dfrac{x-y}{1+xy}+y}{1-\dfrac{(x-y)y}{1+xy}}$

(3) Solve the equations

$$\text{(i.)}\quad \frac{x-mb}{x-ma} = \frac{x-na}{x-nb}$$

$$\text{(ii.)}\quad \frac{1}{x+7} + \frac{2}{x+14} = \frac{3}{x-21}$$

$$\text{(iii.)}\quad \left.\begin{aligned} ax+by&=ab \\ x+y&=a+b \end{aligned}\right\}$$

(4) Find the sum of

$$\frac{x}{x^2-1} + \frac{x^2+x-1}{x^3-x^2+x-1} + \frac{x^2-x-1}{x^3+x^2+x+1} - \frac{x^3}{x^4-1}$$

From your result infer the value of

$$\frac{a}{a^2-b^2} + \frac{a^2+ab-b^2}{a^3-a^2b+ab^2-b^3} + \frac{a^2-ab-b^2}{a^3+a^2b+ab^2+b^3} - \frac{a^3}{a^4-b^4}$$

(5) Divide $a^{16}+\frac{b^{16}}{64}$ by $a^8+\frac{a^4b^4}{2}+\frac{b^8}{8}$

(6) Solve the equations

$$\text{(i.)}\quad \frac{5x+7}{3} + \frac{3x+11}{4} + \frac{3+x}{7} - 15\tfrac{3}{4} = 0$$

$$\text{(ii.)}\quad (x-4)(x+7) = (x+2)(x-14)$$

(7) A man walks a miles an hour for b hours. How far does he get?

A man walks $a+b$ miles an hour. How long would it take him to walk $a^2+2ab+b^2$ miles?

XVIII.

(1) Multiply together $x+a$, $x+b$, $x+c$, $x+d$, and show what the result becomes if $a=b=-1$, $c=d=2$

(2) Divide $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$ by x^3+y^3

and $a^3-\frac{1}{b^3}$ by $a-\frac{1}{b}$

(3) Simplify

(i.) $\dfrac{(x+y)^3-(x-y)^3}{(x^2-y^2)(3x^2+y^2)} \div \dfrac{2y}{x^4-y^4}$

(ii.) $\dfrac{3a+2x}{3a-2x} - \dfrac{3a-2x}{3a+2x} - \dfrac{16x^2}{9a^2-4x^2}$

(iii.) $\dfrac{1}{x^2+3x+2} + \dfrac{2x}{x^2+4x+3} + \dfrac{1}{x^2+5x+6}$

(4) Resolve into factors

(i.) $6x^2-5xy-6y^2$

(ii.) $\dfrac{9x^2}{a^2} - \dfrac{b^2}{16y^2}$

(iii.) $(a+b)^3+(a-b)^3$

(5) If $x = \dfrac{a+b}{1+ab}$, prove that $\dfrac{1+x}{1-x} = \dfrac{(1+a)(1+b)}{(1-a)(1-b)}$

(6) Solve the equations

(i.) $x-3-3(x-5)+5(x-7)=4$

(ii.) $\dfrac{1}{7}\left(x-\dfrac{1}{2}\right) + \dfrac{1}{5}\left(x+\dfrac{2}{3}\right) = \dfrac{43}{30}$

(7) A sum of money is to be divided between two men. The elder gets £243 more than the younger, who received $\frac{5}{12}$ of the whole. What was the amount to be divided?

XIX.

(1) Simplify $3x-[2a-2\{a-(a-1)\}+2]$

Add together $\dfrac{a^2}{3}-\dfrac{b^2}{3}+\dfrac{c^2}{6}$, $\dfrac{a^2}{3}+\dfrac{b^2}{3}-\dfrac{c^2}{3}$, $\dfrac{b^2}{3}-\dfrac{a^2}{3}+\dfrac{c^2}{6}$

and from the sum subtract $\dfrac{a^2}{3}+\dfrac{b^2}{3}-\dfrac{c^2}{3}$

(2) Multiply $3x^3+4x^2y-5xy^2+3y^3$ by $2x-y$

and divide $\dfrac{x^3}{125} + 27$ by $\dfrac{x}{5} + 3$

(3) Simplify

(i.) $(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(c+a-b)$

(ii.) $\dfrac{4a^2x^3}{5cd^4} \times \dfrac{7c^2d}{2ay^4} \times \dfrac{15ad^3y^3}{84cx^4}$

(4) Define the Greatest Common Measure of two algebraical quantities, and obtain that of

$$x^3+3x^2y+3xy^2+y^3 \text{ and } x^3+x^2y+xy^2+y^3$$

(5) Solve

(i.) $\dfrac{5x-7}{2} - \dfrac{2x-7}{3} = 3x-14$

(ii.) $(3x+1)(3x-1)+2x-7=9x^2+16$

(iii.) $\dfrac{1}{9}\left\{3x - 6 - 5\left(\dfrac{7x}{2} - 5\right)\right\} + 13(x-5) + \dfrac{1}{4} = 0$

(6) Find factors of

(i.) $x^2+18x+72$ (ii.) $x^2+12x-108$

(iii.) $9x^2-16y^2$ (iv.) $x^3+15x^2+62x+72$

(7) Write down an expression of the sixth degree which shall leave the same remainder, $3x+1$, whether it is divided by x^2-2x+2, or x^2+2x+2

XX.

(1) Simplify, by removing the brackets, the following expressions

(i.) $3a-[4a-(7a-b)-(2b-3a)-\{5a-7(b-a)\}]$

(ii.) $x^4-[2x^3-\{5x^2-(3x+1)\}]-[x^4-(2x^3+5x^2-3x)]$

(2) What quantity must be added to $3x^2-9x+7$ to obtain $5x^2-8x-9$; and what quantity must be taken from the sum of these to get $2x-3x^2+4$?

(3) Resolve into their simplest factors

$16a^4-x^4$, $a^2+6ab+9b^2$, $a^2-5ax+6x^2$, a^4-5a^2-36

(4) Prove that $a^5 \times a^3 = a^8$, and $a^5 \div a^3 = a^2$,

(5) Determine a value of x which satisfies each of the following equations

(i.) $5(x-3)+2x^2-7=2x(x-3)+8x-1$

(ii.) $\frac{x-7}{2}+\frac{x}{3}-3=\frac{x+1}{10}$

(iii.) $\frac{3x-1}{2x-1}-\frac{4x-2}{3x-2}=\frac{1}{6}$

(6) A and B start at the same time to meet each other from two places 17 miles apart. A walks $3\frac{3}{4}$ and B $3\frac{1}{4}$ miles per hour. Where will they meet?

XXI.

(1) Find the value of

(i.) $p^2+q^2+r^2+2pq+2qr+2rp$
(ii.) $3^p-q^3+r^{r+p}$
(iii.) $p^2q^2+q^2r^2+r^2p^2$
(iv.) $\{(p+q+r)^3\}^2$
(v.) $p^3+q^3+r^3+3p^2(q+r)+3q^2(r+p)+3r^2(p+q)$
$+6pqr$ when $p=1,\ q=-1,\ r=2$

(2) Add up $\frac{1}{6}x-\frac{1}{3}y+\frac{1}{4}z,\ \frac{1}{6}y-\frac{1}{3}z+\frac{1}{4}x,$
$\frac{1}{6}z-\frac{1}{3}x+\frac{1}{4}y,\ \frac{1}{7}x-\frac{1}{8}y+\frac{1}{9}z$

(3) Multiply $1+x^3+x^6$ by $1-3x+3x^2-x^3$
and $1+\frac{x^3}{8}+\frac{x^6}{64}$ by $1-\frac{3x}{2}+\frac{3x^2}{4}-\frac{x^3}{8}$

(4) Divide $x^{15}+x^9+x^8+x^7+x^6+x^4+x^3+x^2+x+1$
by x^6-x^5+1

(5) Divide $\frac{x^4}{16}-1$ by $\frac{x}{2}-1$

(6) Solve

(i.) $2(x-7)-5(x-8)=2$

(ii.) $\frac{x-7}{2}+\frac{x-9}{5}=1$

(7) Write down algebraical expressions for the sum, the product, the sum of the squares, the square of the sum, and the square of the sum of the squares of x and y, and an expression of the ninth degree, which is divisible by each of them.

XXII.

(1) Find the value of

$$8x^3 + 64y^3 - 27z^3 + 72xyz$$

if $x = -1,\ y = 14,\ z = 18$

(2) A man can walk a feet in b seconds. How many miles can he walk in an hour?

(3) Simplify

(i.) $9a - 2[4a - 2\{a - 3(a - \overline{a - a})\}]$

(ii.) $\dfrac{3ab}{4c^2x} \times \dfrac{5c^3y^2}{2a^3} \times \dfrac{16x^4}{45b^4}$

(iii.) $\dfrac{1}{x-1} - \dfrac{2}{x-2} + \dfrac{1}{x-3}$

(iv.) $\dfrac{1}{(a-b)(a-c)} - \dfrac{1}{(a-b)(b-c)} + \dfrac{1}{(a-c)(b-c)}$

(v.) $\dfrac{x^2 - y^2}{x^3 + y^3} \times \dfrac{x^2 + y^2}{x^3 - y^3} \times \dfrac{x^4 + x^2y^2 + y^4}{x^4 - y^4}$

(4) Find the factors of

(i.) $4x^2 - 25y^2$ (ii.) $\frac{1}{4} - y^4$

(iii.) $x^2 + 2x - 360$ (iv.) $x^2 - 26x - 360$

(v.) $x^2 + 30x + 225$ (vi.) $x^3 - 8 + 7(x - 2)$

(5) Solve the equations

(i.) $\dfrac{x}{14} - \dfrac{x}{21} = \dfrac{1}{6}$

(ii.) $2(x - 11) - 5(x - 13) = 10$

(iii.) $\dfrac{x+1}{5x+1} = \dfrac{3x+2}{15x}$

XXIII.

(1) Find the value of

$$(x-2)(x-3)(x-4)(x-5)-5(x-2)(x-3)(x-6)$$
$$-10(x-2)(x-5)-20(x+1)+60$$

when $x=-1$.

(2) Find the product of

(i.) $x^2-3xy+9y^2$ and $x^2+3xy+9y^2$

(ii.) $x^2-\frac{xy}{3}+\frac{y^2}{9}$ and $x^2+\frac{xy}{3}+\frac{y^2}{9}$

(3) Divide (i.) x^4-16y^4 by $2y-x$

(ii.) $x^4-\frac{y^4}{16}$ by $\frac{y}{2}-x$

(4) Find the G.C.M. and L.C.M. of

$a^3-3a^2b+3ab^2-b^3$ and $a^3+a^2b-ab^2-b^3$

(5) Simplify

(i.) $\frac{1}{2(x-2)}-\frac{1}{x-3}+\frac{1}{2(x-4)}$

(ii.) $\frac{1}{(a-b)(x-a)}+\frac{1}{(b-a)(x-b)}$

(6) Solve the equations

(i.) $\frac{x-1}{3}+\frac{x+1}{5}=\frac{x+11}{3}$

(ii.) $2(x-3a)+3(x+2a)=0$

(7) Write down an algebraical expression which shall leave a remainder 1, whether it is divided by 3, by 4, or by 5.

Use it to obtain the numbers below 200 which leave a remainder 1 when divided by 3, by 4, or by 5.

XXIV.

(1) Subtract $p^4+p^2q^2+2p^2q+p-q+q^4$
from $p^4+p^3-p^2q+pq^2+p-q-q^3-q^4$

(2) Multiply together

$$x^2+y^2+z^2,\ x^2-y^2+z^2,\ x^2+y^2-z^2, \text{ and } x^2-y^2-z^2$$

(3) Divide $x^{11}+114x^4+9x^{10}+10+36x^9+77x+85x^8+84x^2+118x^7+117x^3+88x^5+135x^6$ by $7x+5x^2+10+x^3$

and $3+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}$ by $a+b+c$

(4) What is the rule for indices when two powers of the same quantity are multiplied together? And what is the rule when one is divided by the other? Explain the reason of these rules.

(5) Solve the following equations

(i.) $x+\frac{x-19}{3}=\frac{4}{3}-\frac{x}{5}$

(ii.) $\frac{4x+3}{7}+\frac{5x-1}{3x+5}=\frac{20x+29}{35}+\frac{3}{5}$

(6) Show that $(x-y)^2+(x-y)(y-z)+(y-z)^2$

$$=\frac{1}{2}\{(x-y)^2+(y-z)^2+(z-x)^2\}$$

(7) A labourer was employed for 80 working days on condition that for every day he worked he should receive 3s., and for every day he was absent he should forfeit 6d. At the end of the time he had £9 0s. 6d. to receive. Find the number of days he worked.

XXV.

(1) Multiply $6x^3-20x^2+x+36$ by x^2-2x+6, using detached coefficients.

(2) Divide

$$\frac{1}{x}-\frac{a^2-b-c}{x^3}-\frac{a(b-c)}{x^4}+\frac{bc}{x^5} \text{ by } \frac{1}{x^3}-\frac{a}{x^4}+\frac{c}{x^5}$$

(3) Resolve into factors

(i.) $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$

(ii.) $x^2 - (a + b)x + ab$

(iii.) $x^6 - y^6$

(iv.) $x^3 + 16x^2 + 79x + 120$

(4) Find the L.C.M. of

$x^2 - 10x + 24,\ x^2 - 8x + 12,$ and $x^2 - 6x + 8$

and find the G.C.M. of

$3a^3 - 3a^2b + ab^2 - b^3$ and $4a^2 - 5ab + b^2$

(5) Simplify

(i.) $\dfrac{4}{x+1} - \dfrac{3}{(x+1)^2} + \dfrac{1}{(x+1)^3} - \dfrac{4}{x+2}$

(ii.) $\left\{\dfrac{x}{1+x} + \dfrac{1-x}{x}\right\} \div \left\{\dfrac{x}{1+x} - \dfrac{1-x}{x}\right\}$

(iii.) $\dfrac{1}{a - \dfrac{1}{b - \dfrac{1}{c - a}}}$

(6) Solve the equations

(i.) $\dfrac{4+x}{4} + \dfrac{x-1}{7} = 7 - \dfrac{23-x}{5}$

(ii.) $\dfrac{x-4a}{x-3a} + \dfrac{x-5a}{x-4a} = \dfrac{x+6a}{x-4a} + \dfrac{x+5a}{x-3a}$

(iii.) $\dfrac{x-1}{9} + \dfrac{1}{x} = 1$

(iv.) $2x + 6y = 3,\ 6x - 3y = 2$

(7) The sum of the ages of a father and son is half what it will be 25 years hence, and the difference is one-third of what the sum will be 20 years hence; find how old they are respectively.

XXVI.

(1) Simplify

(i.) $\dfrac{1}{1-6x} + \dfrac{1}{1-2x} - \dfrac{2}{1-3x}$

(ii.) $\dfrac{2}{1+3x} - \dfrac{1}{1+2x} - \dfrac{1}{1+6x}$

(iii.) $\dfrac{2x}{1+3x^2} - \dfrac{x}{1+6x^2} - \dfrac{x}{1+2x^2}$

(iv.) $\dfrac{2}{x^2} - \dfrac{1}{1+x+x^2} - \dfrac{1}{1-x+x^2}$

(v.) $\dfrac{5x^4+22x^3+12x^2+23x+28}{5x^4+22x^3+22x^2+17x+42}$

(vi.) $\dfrac{x^4+x^2+1}{x^2+x+1} + \dfrac{x^4+x^2+1}{x^2-x+1}$

(2) Solve the equations

(i.) $\dfrac{x+1}{2} + \dfrac{x+2}{3} + \dfrac{x+3}{4} = 16$

(ii.) $\dfrac{x-7}{11} - \dfrac{3x-5}{7} + \dfrac{125}{77} = 2x-17$

(iii.) $\dfrac{9x+7}{2} - \left(x - \dfrac{x-2}{7}\right) = 36$

(iv.) $\dfrac{x-1\frac{25}{26}}{2} - \dfrac{2-6x}{13} = x - \dfrac{5x - \dfrac{10-3x}{4}}{39}$

(v.) $\cdot15x + \cdot2 - \cdot875x + \cdot375 = \cdot0625x - 1$

(vi.) $\dfrac{3-2x}{1-2x} - \dfrac{5-2x}{7-2x} = 1 - \dfrac{4x^2-2}{7-16x+4x^2}$

(3) Find the factors of

(i.) $(x-1)(1+y)+(1-y)(1+x)$
(ii.) $(x+1)(1-y)-(1+y)(1-x)$
(iii.) $(x-y)(1+yz)+(y-z)(1+xy)$
(iv.) $(x+y)(1-yz)-(y+z)(1-xy)$
(v.) $(x-1)^2-5(x-1)+6$
(vi.) $7-3(x-5)-4(x-5)^2$

(4) Two persons at a distance of 240 leagues set out to meet each other, and travel at the rates of 7 and 8 leagues a day towards one another. When and where will they meet?

XXVII.

(1) Add together $x+y$, $3x-y-3z$, $2y-2x+z$, and multiply the result by $x-y-z$.

Find the value of $\dfrac{x^5-x^4y+x^3y^3}{y^5-y^4x+y^3x^3}$ when $x=2y$, and prove that $(x+y)(x^2+y^2)(x^4+y^4)=\dfrac{x^8-y^8}{x-y}$

(2) Prove that any common factor of two expressions is a factor of the sum or difference of any multiples of them.

What use is often made of this fact in the process of finding the highest common divisor of two algebraic expressions?

Express

$$4x^2-6yz-(9y^2+z^2),\ 9y^2+4xz-(4x^2+z^2),$$
$$z^2-12xy-(4x^2+9y^2)$$

in factors, and hence write down their L.C.M.

(3) Simplify

(i.) $\dfrac{x^6+y^6}{x^6-y^6}\times\dfrac{x-y}{x+y}\div\dfrac{x^4-x^2y^2+y^4}{x^4+x^2y^2+y^4}$

(ii.) $\dfrac{1}{2(x-1)}-\dfrac{x-5}{x^2-7x+10}+\dfrac{1}{2}\cdot\dfrac{x-6}{x^2-9x+18}$

(4) Solve the equations

(i.) $\frac{2x-5}{4} - \frac{x-2}{6} = \frac{x}{7} - \frac{1}{4}$

(ii.) $\begin{cases} 2x+3y=5 \\ 10x-6y=11 \end{cases}$

(5) Resolve into factors

$9a^2-36x^2,\ 4a^2-4ax+x^2,\ 2a^2+3ax-2x^2,$

and write down the L.C.M. of the expressions.

(6) Simplify

$$\left(2x-\frac{x^2-y^2}{x}\right)\left(3y+\frac{x^2+y^2}{y}\right) \div \left(\frac{x^2}{y^2}+5+\frac{4y^2}{x^2}\right)$$

(7) If $a+b+c=0$

show that $a^3+b^3+c^3=3abc$

XXVIII.

(1) Simplify $\frac{a}{a+x} + \left(\frac{x}{a+x}\right)^2 + \frac{ax}{(a+x)^2}$

(2) Multiply x^2+3x-6 by x^3-x^2+1

and divide x^4+4x+3 by x^2-2x+3

(3) Find the G.C.M. and L.C.M. of

$6x^3-11x^2+5x-3,\ 9x^3-9x^2+5x-2$

(4) Reduce $\frac{1}{2a+x} + \frac{1}{2a-x} - \frac{3a}{4a^2-x^2}$

(5) Solve

(i.) $5-3(4-x)+4(3-2x)=0$

(ii.) $\frac{5}{7}\left(x-9\right) + \frac{7}{9}\left(x-5\right) = \frac{9}{5}\left(x-7\right) + 1\frac{2}{3}$

(iii.) $\left.\begin{matrix} 5x+4y=58 \\ 3x+7y=67 \end{matrix}\right\}$

(6) Given that $(2983)^2 = 889829$, find by means of an algebraical formula $(2984)^2$ and $(29831)^2$

(7) Find a number whose excess above 50 is equal three times its defect from 90.

XXIX.

(1) Find the value of

$$(a+b)^3 + (b+c)^3 + (c+a)^3 - 3(a+b)(b+c)(c+a)$$

when $a = -\frac{1}{2}$, $b = -\frac{1}{3}$, $c = .8\dot{3}$

(2) Find the continued product of $x+2$, $x+3$, $x+4$, and $x+5$, using the method of detached coefficients, if possible.

(3) Arrange the product of $x+a$, $x+b$, $x+c$, $x+d$, in the form $x^4 + (P)x^3 + (Q)x^2 + (R)x + S$, where P, Q, R, and S do not contain x.

(4) Divide $1 + x - 17x^8 + 15x^9$ by $1 - 2x + x^2$, using the method of detached coefficients, if possible.

(5) Divide $a + (a+b)x + (a+b+c)x^2 + (a+b+c)x^3 + (b+c)x^4 + cx^5$ by $a + bx + cx^2$

(6) Solve the equations

(i.) $\left.\begin{array}{l} 7(x+2y) + 5(x-2y) = 74 \\ 3(x+2y) - 2(x-2y) = 11 \end{array}\right\}$

(ii.) $15(x+3)^2 + 19(x+3)^2 = 34x^2$

(7) If I add 3 times a certain number to 4 times the next higher number, I get 8 times the next lower number. What is the number?

XXX.

(1) If $a = 2$, $b = 3$, $c = -2$ find the value of

(i.) $3ab - 2ab^2 + b^3$

(ii.) $2ac^2 - 3abc + 5bc^3$

(2) Divide

(i.) $a^4 - a^3b + 2a^2b^2 - ab^3 + b^4$ by $a^2 + b^2$
(ii.) $a^3 + b^3 - 8c^3 + 3a^2b + 3ab^2$ by $a + b - 2c$
(iii.) $x^2 + 1 + \frac{1}{x^2}$ by $x + 1 + \frac{1}{x}$

(3) Simplify

$$2x - [a - \{3b - 5c\} + \{4a - b - x + c\} - \{2b - (3a + 3c)\}]$$

(4) Solve

(i.) $7x + 19 = 5x + 7$
(ii.) $\frac{x-5}{2} - \frac{x-4}{3} = \frac{x-3}{2} - (x-2)$
(iii.) $\left.\begin{array}{l} 3x + 5y = 19 \\ 5x - 4y = 7 \end{array}\right\}$

(5) H.C.F. of $x^2 - 5x + 4$ and $x^3 - 5x + 4$

(6) Value of

$$\frac{1}{x^2 + 5ax + 6a^2} - \frac{2}{x^2 + 4ax + 3a^2} + \frac{1}{x^2 + 3ax + 2a^2}$$

(7) Take the number of years in your age. Add three to it. Square the result. Deduct 9. Divide this result by the number of years in your age. Deduct 6. And there remains your age. Explain this.

XXXI.

(1) Find the value of

(i.) $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$
(ii.) $a^3 + b^3 + c^3 - 3abc$

when $a = -1,\ b = -2,\ c = 3$

(2) Factorize

(i.) $x^2 + (a + b)x + ab$
(ii.) $2x^2 + 5x + 2$
(iii.) $a^4 + a^2b^2 + b^4$
(iv.) $a^6 - b^6$

(3) Find the G.C.M. of

$$x^4 - 2x + 1 \text{ and } x^4 + 2x^3 + 2x^2 - 1$$

Obtain the L.C.M. in factors.

(4) Find the L.C.M. of

(i.) $a^2(x+y)^3z$, $a^3b(x+y)z^2$, and $5a^3b^2z^3$

(ii.) x^2+6x+8, $x^2+7x+12$, and $(x+2)^2(x+3)^3$

(5) Solve the equations

(i.) $\dfrac{x+1}{2} + \dfrac{x+2}{3} = 16 - \dfrac{x+3}{4}$

(ii.) $3{\cdot}25x - 5{\cdot}007 - x = {\cdot}2 - {\cdot}34x$

(iii.) $\begin{array}{l} 2x - 5y = 4 \\ 5x + 3y = 41 \end{array}$

(6) Simplify

(i.) $\dfrac{6ac + 10bc + 9ad + 15bd}{6c^2 + 9cd - 2c - 3d}$

(ii.) $\dfrac{a^3}{(a+b)^3} - \dfrac{ab}{(a+b)^2} + \dfrac{b}{a+b}$

(7) The sum of two numbers is 13, the difference of their squares 39. Find them.

XXXII.

(1) Say what you mean by term, factor, coefficient. Denote by brackets the different terms in $1\frac{1}{2} + 2\frac{1}{3}$ of $\frac{3}{4} + 7\frac{1}{2} - 2\frac{1}{8}$ of 5, and simplify the expression.

(2) Subtract

$$3x - 5y + 3z + y^2 \text{ from } 2x + 3y + 3z + 1$$

(3) Multiply $a^2 - 2ab + 3b^2$ by $a^2 + 2ab + 3b^2$

and divide $x^4 - 16y^4$ by $\dfrac{1}{2}x + y$

(4) Solve the equations

(i.) $5(x-7)-7(x-8)=1$

(ii.) $\dfrac{x+1}{7} - \dfrac{2-x}{21} + \dfrac{3-x}{28} = 0$

(iii.) $\left.\begin{aligned} 8x+3y&=23 \\ 2y-3x&=7 \end{aligned}\right\}$

(iv.) $\dfrac{x-5a}{x+2a} + \dfrac{x-2a}{x+5a} = 2$

(5) One-third of a form are promoted, and 12 new boys taken into it; the form is found to be increased by $\frac{1}{15}$th of its former size. Find the former size.

(6) One side of a right-angled triangle exceeds another by two inches, and the hypothenuse is 10 inches. Find the sides.

XXXIII.

(1) Multiply $\left(a+\dfrac{1}{a}\right)^3 - \left(a^3+\dfrac{1}{a^3}\right)$ by $\dfrac{1}{3}\left(a-\dfrac{1}{a}\right)$

and divide $a^3+b^3+3ab-1$ by $a+b-1$

(2) Resolve into factors $x^2-x-110$, x^4-81, and

$a^2-b^2-c^2+d^2+2(ad+bc)$

(3) Find the highest common divisor of

$3x^3-11x^2+27x-14$ and $3x^3+13x^2-16x+4$

(4) Simplify the expressions

(i.) $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y}{x}-2}$ (ii.) $\dfrac{1+\dfrac{x}{y}+\dfrac{x^2}{y^2}}{\dfrac{x}{y}-\dfrac{y^2}{x^2}}$

(5) Solve the equations

(i.) $\dfrac{1}{x-3} - \dfrac{1}{x-5} = \dfrac{1}{x-2} - \dfrac{1}{x-4}$

(ii.) $\dfrac{x+1}{2x-1} - \dfrac{2-x}{4x} = \dfrac{3}{4}$

(6) Show that

$$(x+1)(x+2)(x+3)(x+4)+1=(x^2+5x+5)^2$$

and hence factorize

$$(x+1)(x+2)(x+3)(x+4)-(y+1)(y+2)(y+3)(y+4)$$

(7) Two rectangular fields were supposed to contain each 4 acres, but on measurement it was found that one contained 140 square yards more, the other 160 square yards less than that area, and that one was 10 yards longer and 10 yards narrower than the other. Find their dimensions.

XXXIV.

(1) Find the numerical value of

$$\frac{3(b-2c)+\{5a+3b-3\sqrt{a^4+b^2c^2}\}}{a-b+c+2\sqrt[5]{b^3+3c^3-a^3}}$$

when $a=-2,\ b=3,\ c=-1$

(2) Illustrate geometrically the identities

(i.) $(a+b)^2=a^2+2ab+b^2$

(ii.) $(a+b)(a-b)=a^2-b^2$

(3) Find the lowest common multiple of

$$4x^2(x-a),\ 2x(2x+2a)^2,\ 6(a^3+x^3)$$

and the highest common divisor of

$$(x-y)^5-x^5+y^5 \text{ and } (x-y)^3-x^3+y^3$$

(4) Simplify the expressions

(i.) $$\frac{x}{x-1}-\frac{x^3}{1-x^2}-\frac{x^2+1}{x^2-2x+1}+\frac{x+1}{(x-1)^2}$$

(ii.) $$\frac{1+x^3}{1-\dfrac{x}{1+\dfrac{x}{1-x}}}-\frac{1-x^3}{1+\dfrac{x}{1-\dfrac{x}{1+x}}}$$

(iii.) $$\frac{3x-x^3}{1-3x^2}-\frac{2x}{1-x^2}\div\left\{1-\frac{2x(x^3-3x)}{1-4x^2+3x^4}\right\}$$

(5) Solve the equations

(i.) $$\frac{3x+5}{7} + \frac{1}{2} = \frac{5x+7}{11} - \frac{x-4}{2}$$

(ii.) $$\left.\begin{aligned} ax - by &= b^2 - a^2 \\ \frac{x}{a} + \frac{y}{b} &= -2 \end{aligned}\right\}$$

(6) A room has an area of 224 square feet; by deducting $\frac{1}{8}$ from its length and 2 feet from its breadth, its area is reduced 25 per cent. Find its dimensions.

XXXV.

(1) Find the value of

$$\frac{a-b}{5a+5b} + \frac{a^2-(b-1)^2}{119b} + (3a-b-1)^2 - \cdot 06\dot{7}$$

when $a = \frac{1}{3}$, $b = \frac{1}{5}$

(2) Divide

$$x^3 + 2bx^2 - (a^2 - ab - b^2)x - a^2b + ab^2 \text{ by } x - (a-b)$$

(3) Prove that

$$(x+1)^3 + (x-2)^3 + (1-2x)^3 = 3(x+1)(x-2)(1-2x)$$

(4) Reduce to its lowest terms

$$\frac{2x^4 + 5x^3 + 3x^2 + x - 2}{2x^4 - x^3 - 3x^2 - 5x - 2}$$

(5) Solve the equations

(i.) $$\frac{1}{6}\left(x+3\right) - \frac{1}{9}\left(2-x\right) = \frac{2}{3}\left(x-4\right) - \frac{5}{63}\left(16-x\right)$$

(ii.) $$7(x-5) - (y-2) = 3(4y-3) - (x+10) = 0$$

(6) Find the factors, given that $x - 2a - 3b$ is one of them, of

$$x^3 - 6ax^2 - 8a^3 - 9bx^2 + 12a^2x + 27b^2(x-b)$$
$$+ 36ab(x-a) - 54ab^2$$

(7) One quarter of the length of a shelf is filled with books 2 inches thick; 9 inches of the shelf with books 3 inches thick; and the remainder with books 1 inch thick. There are 50 books in all; find the length of the shelf.

XXXVI.

(1) Simplify

(i.) $\{2x-(5y-\overline{3z+7})\}-[4+\{x-(3y+2z+5)\}]$

(ii.) $\left(1-\frac{b^2}{a^2}\right)\left(1-\frac{ab-b^2}{a^2}\right)\frac{a^4}{a^3+b^3}\cdot\frac{a-b}{a^2+b^2}$

(iii.) $$\cfrac{1}{x^2-\cfrac{x^3+1}{x+\cfrac{1}{x-1}}}$$

(2) What value of x will make the sum of

$$\frac{x-a}{2(a-b)} \text{ and } \frac{x+5b}{3(a+b)} \text{ equal to 2?}$$

(3) Find the L.C.M. of

$$8(a-x),\ 4x^2(a^3-x^3),\ 6a^2(a^2-x^2),$$

and the G.C.M. of

$$a^3-2a^2+3a-6,\ a^4-a^3-a^2-2a$$

(4) Solve the equations

(i.) $\cdot5x+\cdot6x+\cdot8=\cdot75x+\cdot25$

(ii.) $\left.\begin{array}{l}3x-2y=19\\2x-3y=43\end{array}\right\}$

(iii.) $\left\{\begin{array}{l}\frac{1}{x}+\frac{1}{y}=a\\[2mm]\frac{1}{x}+\frac{1}{z}=b\\[2mm]\frac{1}{y}+\frac{1}{z}=c\end{array}\right.$

(5) Find the square of $3a^3 - 5a^2b + 6ab^2 - 2b^3$, and the cube of $3x^3 - 5x^2 - 2$.

(6) Find factors of

(i.) $4(a+b+c)^2 - (a+b-c)^2$

(ii.) $(x+2)^2 - 2(x+2)(y+3) + (y+3)^2$

(7) A rectangular grass plot, 8 ft. longer than it is broad, is surrounded by a path 2 ft. 6 in. wide. The cost of making the path, at 1s. 6d. a square yard, is £3 2s. 6d. Find the length and breadth of the plot of grass.

XXXVII.

(1) Find the numerical value of

$$\frac{2a - [b - \{a - (b - c)\} - \sqrt{c^4 + a^2b^2 + 2a^3c}]}{\sqrt{2a^2 + b^2 + c^2}}$$

when $a = -1,\ b = 5,\ c = 3$

(2) One of the factors of $x^3 + y^3 + x^2 - xy + y^2$ is $x + y + 1$, find the other factor.

Divide

$$x^3 + y^3 - 3xy + 1 \text{ by } \tfrac{1}{2}[(x-y)^2 + (x-1)^2 + (y-1)^2]$$

(3) Find the highest common divisor of

$$x^4 - 2y^4 + x^2y^2 + 3x^3y - 3xy^3 \text{ and } 2x^3 + 2y^3 + 7x^2y + 7xy^2$$

and the least common multiple of

$$a^2 - b^2 - ac + bc,\ a^2 + b^2 + 2ab - ac - bc,$$

and $a^2 + b^2 - c^2 + 2ab$

(4) Simplify the expressions

(i.) $1 + \dfrac{1}{x+a} - \dfrac{1}{(x+a)(x+a+1)}$

(ii.) $\dfrac{x^4 + x^2y^2 + y^4}{x^2 + xy + y^2} \times \dfrac{x^2 + y^2}{x^4 + y^4 + 2x^2y^2 - x^3y - xy^3}$

(iii.) $\dfrac{(a+b+c)^3 - (b+c-a)^3 - 24abc}{(c+a-b)^3 + (a+b-c)^3}$

(5) Solve the equations

$$\text{(i.)}\quad \frac{7}{2}x - 1 - \frac{3x+1}{2} = \frac{4x+51}{14}$$

$$\text{(ii.)}\quad \left.\begin{aligned} \frac{9}{10}x - \frac{8}{9}y &= \frac{11}{6} \\ \frac{10}{9}x + \frac{9}{8}y &= \frac{643}{72} \end{aligned}\right\}$$

(6) Express n and n^3 each as the difference of two squares.

(7) Four men, A, B, C, and D, paid for building a ship which cost £2607; B paid twice as much as A, C paid as much as A and B, and D paid as much as C and B. What did each pay?

XXXVIII.

(1) Find the numerical value of

$$\frac{3a - [2(b+c) - \sqrt{a^2+b^2+c^2+ac}]}{a^2+b^2+c^2-\sqrt{a^4+b^2c^2}}$$

when $\quad a = -2,\ b = 3,\ c = -1$

(2) Multiply

$64x^2 + 9y^2 + 25z^2 + 15yz - 40zx + 24xy$ by $8x - 3y + 5z$

Divide $\quad 1 + a^5 + a^{10}$ by $a^2 + a + 1$

(3) Find the highest common divisor of

$$x^4 - 3x^3 - 2x^2 + 12x - 8 \text{ and } x^3 - 7x + 6,$$

and the lowest common multiple of

$$x^2 - 36y^2,\ x^2 + 7xy + 6y^2,\ x^2 + 5xy - 6y^2$$

(4) Simplify the expressions

$$\text{(i.)}\quad \frac{a+2}{a} - \frac{a}{a+2} - \frac{a^3 - 2a^2}{2a^2 - 8}$$

$$\text{(ii.)}\quad \frac{1}{x-1} + \frac{2}{x+1} + \frac{3x-2}{1-x^2} - \frac{1}{(x+1)^2}$$

(iii.) $\dfrac{b-c}{a^2-(b-c)^2} + \dfrac{c-a}{b^2-(c-a)^2} + \dfrac{a-b}{c^2-(a-b)^2}$

(iv.) $(1+a)^2 \div \left\{ 1 + \dfrac{a}{1-a+\dfrac{a}{1+a+a^2}} \right.$

(5) If $a+b+c=0$, show that

$$(a+b)^3+(b+c)^3+(c+a)^3=3(a+b)(b+c)(c+a)$$

(6) Solve the equations

(i.) $\frac{1}{5}(x-5)+\frac{1}{3}(x-3) = \frac{1}{12}(5x-3)$

(ii.) $\left.\begin{array}{l} a(x-a)=b(y-b) \\ a(y+a)=b(b-x) \end{array}\right\}$

(7) Ten years ago a father was seven times as old as his son; two years hence twice his age will be equal to five times his son's. What are their present ages?

XXXIX.

(1) Divide $9a^2+3ab-2bc-4c^2$ by $3a-2c$

(2) Write in algebraical symbols: The product of the sum and difference of any two numbers is equal to the difference of their fourth powers divided by the sum of their squares.

Prove that the statement is true.

(3) State and prove the rule for finding the H.C.F. of two algebraical expressions.

Find the H.C.F. of

$x^4+3x^3+3x^2-3x-4$ and $3x^5+2x^4-4x^3-3x^2+x+1$

(4) Simplify

$$\text{(i.)}\quad \left(a+\frac{1}{\frac{1}{b}-\frac{1}{a}}\right)\left(a-\frac{1}{\frac{1}{a}+\frac{1}{b}}\right)\times\frac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$$

$$\text{(ii.)}\quad \frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2-ca}{(b-c)(b-a)}+\frac{c^2-ab}{(c-a)(c-b)}$$

(5) Solve the equations

$$\text{(i.)}\quad \frac{3x-5}{4x-6} = 2-\frac{3x-7}{2x-3}$$

$$\text{(ii.)}\quad \frac{3}{x-2}+\frac{4}{x-3} = \frac{3}{x-4}$$

$$\text{(iii.)}\quad \left.\begin{aligned}\frac{x-y}{3}+\frac{x+y}{5} &= 4\\ \frac{x-y}{5}-\frac{x+y}{9} &= 1\tfrac{11}{45}\end{aligned}\right\}$$

(6) Show that if we take any five consecutive numbers the product of the first and fifth is greater than the product of the second and fourth, and greater still than the square of the third.

(7) On a certain farm one-tenth of the stock are pigs, one-eighth are calves, and there are besides 60 sheep and 33 cows. How many animals are there altogether?

XL.

(1) If $a=1$ and $b=2$, find the numerical value of

$$\sqrt{(a^2-2ab+b^2)}\times\sqrt[3]{(a^3-3a^2b+3ab^2-b^3)}$$

(2) Simplify

$$a-[2b-\{3c+3a-(a-b)\}+\{2a-(b-c)\}]$$

(3) Multiply $3a^2+ab-b^2$ by $a^2-2ab-3b^2$

(4) Divide

(i.) $a^2x^4 - (2ac + b^2)x^2 + c^2$ by $ax^2 - bx - c$

and (ii.) $\frac{x^3}{6} - \frac{x}{4} - \frac{1}{8} + \frac{5x^2}{36}$ by $\frac{x}{3} + \frac{1}{2}$

(5) Find the G.C.M. of

$x^2 - xy - 12y^2$, and $x^2 + 5xy + 6y^2$

and the L.C.M. of

$16x^2 - 1$, $4x^2 - 7x - 2$, and $4x^2 + 7x - 2$

(6) Solve the equations

(i.) $\frac{45}{2x+3} = \frac{57}{4x-5}$

(ii.) $\frac{1}{x-a} - \frac{1}{x-b} = \frac{a-b}{x^2-ab}$

(iii.) $\frac{x+y}{3} + x = 15$, and $\frac{x-y}{5} + y = 6$

(7) A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find the ages of A and B.

XLI.

(1) If $a=6$, $b=5$, $c=4$, $d=9$, find the numerical value of

$$\sqrt{a^2 + b^2 + \frac{d}{3}} + 6\sqrt{4d}$$

(2) Divide $y^8 - 2b^4y^4 + b^8$ by $y^3 + by^2 + b^2y + b^3$

(3) Simplify $\frac{b+y}{b-y} + \frac{4by}{b^2-y^2} + \frac{b-y}{b+y}$

(4) Find the H.C.F. of

$15a^4 + 10a^3b + 4a^2b^2 + 6ab^3 - 3b^4$

and $6a^3 + 19a^2b + 8ab^2 - 5b^3$

(5) Show that

$$(a-2b+c)^2+(a-2b+c)(b-2c+a)+(b-2c+a)^2$$
$$=(b-2c+a)^2+(b-2c+a)(c-2a+b)+(c-2a+b)^2$$

and write down another expression of the same form equal to each of these.

(6) Solve the following equations

(i.) $\frac{1}{2}\left(x-2\right) - \frac{1}{3}\left(x-3\right) + \frac{1}{4}\left(x-4\right) = 4$

(ii.) $a(2x-a)+b(2x-b)=2ab$

(iii.) $6x-5y=11, \ 7y-4x=11$

(7) A carriage, horse, and harness are together worth £144; the carriage is worth $\frac{4}{5}$ of the horse's value, and the harness $\frac{3}{5}$ of the difference between the values of the horse and carriage. What is the value of each?

XLII.

(1) From $(a+1)x+(a-1)y$ take $(a-1)x-(a+1)y$, and multiply $a+2b+3c$ by $a+2b-3c$

(2) Reduce to its simplest form the expression

$$(x+1)(x+2)+(x-1)(x-2)+(x+1)(x-1)$$
$$+(x+2)(x-2)$$

(3) Divide x^4-x^2-2x-1 by x^2+x+1, and $x^6-2x^5-5x^4+20x^3-25x^2+14x-3$ by x^2+2x-3

(4) Find the G.C.M. of

(i.) x^2-5x+6, x^2+x-6, and x^2-6x+8

(ii.) $3x^3-x^2-3x+1$ and $6x^3+7x^2-1$

and the L.C.M. of

x^2-3x+2, x^2+x-2, and x^2-1

(5) Simplify the following expressions

(i.) $\frac{x+1}{x-1}+\frac{1}{x+1}-\frac{3x}{x^2-1}$

(ii.) $\frac{x^3-x^2-4x+4}{x^3-3x^2+6x-4}$

(iii.) $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right)\left(\frac{x^2-1}{x^6-1}\right)$

(iv.) $\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}+\frac{1}{(x-3)(1-x)}$

(v.) $\frac{x^4+x^2+1}{x^2+x+1}-\frac{(x^6-1)(x+1)}{(x^2-x+1)^2(x^2+2x+1)}+\frac{(x-1)^3}{x^2-1}$

(6) Determine x and y from the following equations

(i.) $\left.\begin{array}{l}\cdot 3x+\cdot 2y=5\\ \cdot 5x-\cdot 3y=3\end{array}\right\}$

(ii.) $\left.\begin{array}{l}\frac{7}{x}-\frac{4}{y}=3\\ \frac{2}{3x}+\frac{16}{y}=\frac{7}{3}\end{array}\right\}$

(iii.) $\left.\begin{array}{l}\frac{x-1}{5}-\frac{2y-3}{7}+\frac{x}{6}=1\\ 5x=6y\end{array}\right\}$

(7) A has three times as much money as B, and twice as many coins (which are all half-crowns or shillings); A has 28 half-crowns, and B has 9 shillings. How many shillings has A?

XLIII.

(1) Simplify

(i.) $4\left\{a-\frac{3}{2}\left(b-\frac{4c}{3}\right)\right\}\left\{\frac{1}{2}(2a-b)+2(b-c)\right\}$

(ii.) $\frac{a}{b}-\frac{a^2-b^2}{b}x+\frac{a(a^2-b^2)}{b(b+ax)}x^2$

(2) Find the G.C.M. of

$x^5+11x-12$

and x^5+11x^3+54

(3) Find the L.C.M. of

$$8a^3 - 18ab^2 : 8a^3 + 8a^2b - 6ab^2 : 8a^2 + 2ab - 3b^2$$

and factorize $6x^2 + 10 - x\left(4y + \frac{15}{y}\right)$

(4) Divide $x^4 + 5ax^3 - (n^2 + n - 7)a^2x^2 - 5na^3x - 6a^4$
by $x^2 - (n-2)ax - 2a^2$

(5) Solve

(i.) $\frac{3}{x} + 5\left(\frac{1}{x} + 2\right) = \frac{6}{x} + 1$

(ii.) $$\left.\begin{aligned} x + \frac{3}{y} &= \frac{7}{2} \\ 3x - \frac{2}{y} &= \frac{26}{3} \end{aligned}\right\}$$

(6) Simplify

$$2(81x^4 + y^4) - 2(9x^2 + y^2)(3x - y)^2 + (3x - y)^4,$$

and find the square root of the result.

(7) A wife's age was two-thirds her husband's at marriage. A year after marriage a son was born, and when he was eleven years old, the age of the father was equal to the sum of the ages of the mother and son. What was the age of the father at marriage?

XLIV.

(1) Find the value of

$$(3 - x)(y - 2) - (1 - x)(2y + 5) + (3 + 2x - y)^2$$

when $x = 2$, $y = -1$.

(2) Simplify

(i.) $(a+b)^2 - 2(a^2 + b^2) - (a+b)(a-b)$

(ii.) $\frac{b}{(a-b)(b-c)} + \frac{c}{(b-c)(c-a)} + \frac{a}{(a-c)(a-b)}$

(iii.) $\left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{a+b}{a} + \frac{a-b}{b}\right) \div \left(\frac{a}{b^2} - \frac{b}{a^2}\right)\left(\frac{1}{a} + \frac{1}{b}\right)$

(3) Find the G.C.M. of

$$12x^4 - 30x^3 + 12x^2, \text{ and } 12x^3 - 24x^2 + 16x - 32$$

(4) Find, in the simplest way you can, the product of

$$(a+x+b+y)(a+b-x-y)(a^2+b^2+x^2+y^2+2ab+2xy)$$

(5) Factorize $a^3 + 2a^2b + 2ab^2 + b^3$, and hence show that

$$(x-y)^3 + 2(x-y)^2(y-z) + 2(x-y)(y-z)^2 + (y-z)^3$$
$$= (x-z)\{(y-z)^2 + (y-z)(z-x) + (z-x)^2\}$$

(6) Solve the equations

(i.) $2(x-2) - 3(x+3) + 4(x-4) = 1$

(ii.) $\dfrac{3x}{7} - \dfrac{5x-11}{12} = \dfrac{3x-7}{14}$

(iii.) $\begin{cases} \dfrac{3x-2}{5} - 3 = \dfrac{2y-16}{3} \\ \dfrac{5y-x}{8} = 2 - \dfrac{2y-5x+1}{16} \end{cases}$

(7) A fruiterer sold for 19s. 6d. a certain number of oranges and apples, of which the latter exceeded the former by 180. He sells the apples at the rate of five for 3d., and fifteen oranges bring him in $1\frac{1}{2}$d. more than 35 apples. How many are there of each kind of fruit, and what did the oranges sell at apiece?

XLV.

(1) Simplify

$$(x+y+z)^2 - (y-x)(z-x) - (z-y)(x-y) - (x-z)(y-z)$$

(2) Subtract $\dfrac{a^2+b^2}{a-b}$ from $\dfrac{a^3-b^3}{a^2+b^2}$

and $\dfrac{ab-a^2}{ab+b^2}$ from $\dfrac{ab-b^2}{ab+a^2}$

Multiply together your two results.

(3) Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$
by $a^2(b-c)+b^2(c-a)+c^2(a-b)$
What are the factors of the divisor?

(4) Show that if A is a measure of B and C, it is also a measure of pB ± qC.

Find the H.C.M. of
$2x^3-x^2-x-3$ and $6x^4-11x^3+5x^2+x-6$

(5) Simplify

$$\frac{\frac{x}{z}\left(\frac{x}{y}-\frac{z}{x}\right)+\frac{y}{x}\left(\frac{y}{z}-\frac{x}{y}\right)+\frac{z}{y}\left(\frac{z}{x}-\frac{y}{z}\right)}{\frac{1}{x}\left(\frac{x}{y}-\frac{y}{z}\right)+\frac{1}{y}\left(\frac{y}{z}-\frac{z}{x}\right)+\frac{1}{z}\left(\frac{z}{x}-\frac{x}{y}\right)}$$

(6) Solve the equations

(i.) $(b+c)(x-a)-(c+a)(x-b)=(a+b)(x-c)$

(ii.) $\left.\begin{array}{l}\frac{1}{5}(x-2)=\frac{1}{4}(1-y)\\ 26x+3y+4=0\end{array}\right\}$

(7) In a certain year an omnibus company carried four million fewer passengers than in the preceding year, but the average fare per passenger having been raised from $1\frac{1}{2}$d. to $1\frac{3}{4}$d., its receipts were increased by £37,500. Find the number of passengers carried during each year.

XLVI.

(1) Multiply
$a^3-a^2x+ax^2-x^3$ by $a^3+a^2x+ax^2+x^3$

Simplify
$3\{a-4(b-c)\}+4\{b-3(c-2a)\}-5\{2c+(b-2a)\}$

(2) Divide
$a^4+b^4+c^4-2(a^2b^2+b^2c^2+c^2a^2)$ by $a^2-2ab+b^2-c^2$

(3) Factorize $9x^2-16y^2$, $a^2+6ab+9(b^2-c^2)$,
$(a+b)^2-(a-b)^2$, $27x^3-y^3$, x^6-y^6

(4) Solve

(i.) $3x - \frac{x-2}{5} = \frac{4}{15} - \frac{2}{3}\left\{x - 7\left(5 - \frac{x}{2}\right)\right\}$

(ii.) $3x + 2y = 19$; $\frac{5x}{3} + \frac{x-2}{4} = \frac{y+18}{7} + \frac{7x+y}{4}$

(5) Solve $3x + 5 = 5x - 3$, and justify each step by a reference to Euclid's Axioms.

(6) By what must $(x^2 - 1)(x^2 + x - 2)$ be multiplied so that the result may be a perfect square?

(7) In a building of four flats, the height of each flat was 3 feet more than half the height of the flat below. The highest flat was half the height of the lowest. Find the height of each flat.

XLVII.

(1) Remove the brackets from

$$a^2 - [(b-c)^2 - \{c^2 - (a-b)^2\}]$$

and find the value of

$$x^3 + (p-3)x^2 + (q-3p)x - 3q \text{ when } x = 3$$

(2) Divide the product of

$y^6 - b^2y^4 + b^4y^2 - b^6$ and $y^6 + b^2y^4 + b^4y^2 + b^6$ by $y^4 - b^4$

and $a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2)$ by $a^2(b-c) + b^2(c-a) + c^2(a-b)$

(3) Find factors of

(i.) $x^3 + 12x^2 + 47x + 60$

(ii.) $x^3 - 91x - 330$

Find the G.C.M. of

$x^7 - 3x^6 + x^5 - 4x^2 + 12x - 4$ and $2x^4 - 6x^3 + 3x^2 - 3x + 1$

(4) Simplify $\frac{a+b}{b} - \frac{2a}{a+b} + \frac{a^2b - a^3}{a^2b - b^3}$

If $x = \dfrac{4ab}{a+b}$, find the value of

$$\frac{x-2a}{x+2b} - \frac{x+2a}{x-2b} - \frac{16ab}{4b^2-x^2}$$

(5) Solve the equations

(i.) $$\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}$$

(ii.) $$x - \frac{3x-2y}{7} = y - \frac{4x-5y}{19}$$

$$y - \frac{2x-7y}{6} = 2(x-1) - \frac{3x-7}{8}$$

(6) The adjacent sides of a rectangle contain a and b units of length respectively. Find the area of another rectangle of the same perimeter as the first one, one of whose sides contains $a+c$ units of length.

(7) Two persons, 3750 yds. apart, setting out at the same time are together in 25 minutes if they walk in the same direction, but in 15 minutes if they walk in opposite directions; find their rates of walking.

XLVIII.

(1) When $a=4$, $b=-2$, $c=\frac{3}{2}$, $d=-1$, find the numerical values of

(i.) $$a^3-b^3-(a-b)^3-11(3b+2c)\left(2c^2-\frac{d^2}{2}\right)$$

(ii.) $$\sqrt[3]{4c^2-a(a-2b-d)} \div \sqrt[3]{b^4c+11b^3d^2}$$

(2) Find the highest common divisor, and the lowest common multiple of the three expressions

$$a(a+c)-b(b+c),\quad b(b+a)-c(c+a),\quad c(c+b)-a(a+b)$$

(3) Simplify the expressions

(i.) $$\frac{2}{x^2+x} + \frac{2x-1}{x^2-x+1} - \frac{2x^3-1}{x^4+x}$$

(ii.) $$\left(\frac{x+1}{x-1} + \frac{5}{x-7}\right)\left(\frac{x-2}{3x-5} - \frac{2}{x+3}\right)$$

(iii.) $$\cfrac{3}{2x-3+\cfrac{3}{1-\cfrac{x}{x-6}}}$$

(iv.) $$\cfrac{1}{\cfrac{2}{3-x}-x} - \cfrac{2}{\cfrac{3}{4-x}-x} + \cfrac{1}{\cfrac{6}{5-x}-x}$$

(v.) $$\left\{x + \frac{(9x-8y)y^2}{x^2-9y^2}\right\} \div \left\{x - y + \frac{7y^2}{x+3y}\right\}$$

(4) Divide $9(1-a)(x-x^2)+6ay(1-x)-6x^2y(1-a)$ $-4axy^2$ by $2ay+3x-3ax$

(5) If $x^2+px+40$ and x^2-x-20 have a common factor, find the possible values of p.

(6) Solve the equations

(i.) $$\frac{x+8}{2x+9} - \frac{x-1}{3x-8} = \frac{1}{6}$$

(ii.) $$\begin{cases} 3 - \cfrac{7+\cfrac{2x}{y}}{5} = 5 - \cfrac{5x+9}{3y} \\ y - \cfrac{4+15y}{6x-2} = \cfrac{2xy-13\frac{3}{8}}{2x+5} \end{cases}$$

(7) If $p^2=q^2\text{N}+1$, show that $(2p^2-1)^2=(2pq)^2\text{N}+1$

XLIX.

(1) Simplify

$$x(y-z)-[xy-y(z+a-x)+z(y+a-x)-a(x+z-y)]$$

(2) Multiply $a^3 - 2a^2b + 3ab^2 - b^3$ by $2a^2 + ab - 4b^2$
Divide $(x^3 - y^3)^2 - (y^3 - z^3)^2$ by $x^3 - z^3$

(3) Find the G.C.M. of $4y^2(x^5 - y^5)$ and $2xy(x^3 - y^3)$

(4) Resolve into factors

(i.) $a^2 - 39a + 140$ (ii.) $a^2 + b^2 - 64 + 2ab$

(iii.) $a^6 - 2a^3b^3 + b^6$

(5) Simplify

(i.) $$\frac{1}{a^2 + 3ab + 2b^2} + \frac{3}{a^2 - ab - 2b^2} - \frac{4}{a^2 - 4b^2}$$

(ii.) $$\frac{\dfrac{c-d}{f} - \dfrac{c-f}{d}}{\dfrac{c-d}{f} - \dfrac{d+f}{c}} \times \frac{\dfrac{c}{f} + 1}{1 - \dfrac{f}{d}}$$

(6) Solve the equations

(i.) $$\frac{x-3}{16} - \frac{x-2}{18} = \frac{3}{8}$$

(ii.) $$\frac{x-a}{a+b} - \frac{x+a}{a-b} = \frac{2x}{b}$$

(iii.) $$\begin{aligned} 3x - 5y &= 8 \\ 5x + 3y &= 2 \end{aligned}$$

(7) Divide £24 between two persons so that one may receive half-a-crown more than half the sum the other receives.

L.

(1) Multiply $3\,[5x - 4\,(x - 2y) + 3\,(x - 4y)]$
by $2\,[3(x - 2y) - 2\,\{(x - 4y) + 2\,(x + y)\}]$

(2) Divide $64x^4 + 81$ by $8x^2 - 12x + 9$

(3) Resolve into factors

(i.) $x^2 - 20x + 96$ (ii.) $x^2 - 20x - 96$

(iii.) $x^3 + 8y^3$ (iv.) $a^2 - c^2 - bc + ab$

(4) Solve the equations

(i.) $\cdot\dot{3}(x - 12) + \cdot 25(3x - 37) = \cdot 375(x - 7)$

(ii.) $$1 + \frac{x+y}{7} - \frac{2x+y}{4} = \frac{x-y}{8} - \frac{4x-y}{5} + 3 = 0$$

(iii.) $a(x - a) + c(x + o - b) = b(x - b + c)$

(5) Two consecutive numbers are taken: prove that four times the square of their product plus the square of their sum is a perfect square.

(6) Simplify

(i.) $$\frac{2}{x^2 - 4x + 3} - \frac{1}{x^2 - 3x + 2} - \frac{1}{x^2 - 5x + 6}$$

(ii.) $$\frac{2x^3 - x^2 - 17x + 21}{6x^3 - 7x^2 - 17x + 21}$$

(7) If, in one day, $(l^2 - m^2)$ men earn as much as $(l - m)$ boys, and $(l^2 - lm + m^2)$ boys as much as $(l + m)$ women, how many women will earn as much as $(l^3 + m^3)$ men?

LI.

(1) Subtract $3\{x + 2y - \frac{2}{3}(y - 4)\} - 4(3x - y + 2)$

from $3x + 2 - [7y - 4x - \{4(x - 2) - 3(2y + 1)\}]$

(2) Multiply $-6a^4xy^2$ by $10ax^2y^3$, and divide the product by $-15a^3xy^4$. Verify the result by substituting $a = -1$, $3x = 2$, and $y = -3$, in the multiplicand, multiplier, divisor, and quotient.

(3) Find the greatest common measure of

$4x^4 - 4x^3 + x^2 - 1$ and $2x^3 + 5x^2 - 2x + 3$,

and write down the expressions in factors.

(4) Reduce the following to their simplest forms

(i.) $\frac{1}{a-2} + \frac{1}{a+2} - \frac{4}{a^2-4}$

(ii.) $\frac{x^2-8x+7}{x^2-6x-7}$

(iii.) $\frac{(m-a)(m+a)+n(2m+n)}{m+n+a}$

(iv.) $(a+b+c)^3+(a-b-c)^3+(b-a-c)^3+(c-a-b)^3$

(v.) $\frac{x^2-(y-z)^2}{(z+x)^2-y^2} + \frac{y^2-(z-x)^2}{(x+y)^2-z^2} + \frac{z^2-(x-y)^2}{(y+z)^2-x^2}$

(5) Find a in order that the following may be an identity

$$\frac{1}{(x-1)(x-2)^2} = \frac{1}{(x-2)^2} + \frac{a}{x-2} + \frac{1}{x-1}$$

(6) Find the fraction which is equal to $\frac{3}{7}$ when 10 is added to its numerator, and which is equal to $\frac{1}{3}$ when 4 is subtracted from its denominator.

(7) A man has 81 coins, some of them crowns and the rest shillings; if he exchanged the crowns for florins and the shillings for half crowns he would neither gain nor lose; how many crowns had he?

LII.

(1) Find the value, when $a=\frac{1}{4}$, $b=\frac{1}{5}$, of

$$\frac{a}{b} - \sqrt{\frac{1+a}{1-b}} + \frac{a^2+b^2+ab}{a^2+b^2-ab}$$

(2) Multiply

$$\frac{x^2}{a^2} + \frac{x}{a} + 1 + \frac{a}{x} + \frac{a^2}{x^2} \text{ by } x^2-2ax+a^2$$

(3) Reduce

$$\frac{x^3-3x^2+7x-21}{2x^4+19x^2+35} \text{ to its lowest terms.}$$

(4) Simplify

(i.) $\frac{1}{4}\left\{a-5(b-a)\right\}-\frac{1}{2}\left(b-\frac{a}{3}\right)+\frac{1}{3}\left\{a+\frac{3}{4}(b-2a)\right\}$

(ii.) $\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}$

(iii.) $\dfrac{3}{2x^2+x+\dfrac{1-x}{1+\dfrac{x+1}{3-x}}}$

(5) Solve the equations

(i.) $\frac{1}{14}\left(3x+\frac{2}{3}\right)-\frac{1}{7}\left(4x-6\frac{2}{3}\right)=\frac{1}{2}\left(5x-6\right)$

(ii.) $\left.\begin{aligned}\frac{x+y}{3}-2y&=2\\ \frac{2x-4y}{5}+y&=\frac{23}{5}\end{aligned}\right\}$

(6) I spend half of a certain sum of money in buying wheat at 28s. a quarter and the other half in buying barley at 24s. a quarter. If the wheat had cost 32s. 3d. and the barley 26s. 8d. a quarter, I should altogether have spent £13 5s. more. How many quarters of each did I buy?

LIII.

(1) Simplify the following expressions, and find their product

$5x-(2y-x-1)-2\{3y-2(x+y)\}$
and $3x-4y-2(x-2y)+\frac{3}{2}\{2x-1-3(x-y)\}$

(2) Explain why $(a^2)^3=(a^3)^2=a^6$

If $2^{3x}\times 3^{2x}=5184$, what is the value of x?

Simplify $(a^2b)^2\times(ab^2)^3\div(a^2b^2)^4$

(3) Write down the following expressions in factors, and find their least common multiple

$$4x^2+8x-12,\ 9x^2-9x-54,\ 6x^4-30x^2+24$$

(4) Reduce the following expressions to their simplest forms

(i.) $\dfrac{x+2y}{x-2y} - \dfrac{x-2y}{x+2y}$

(ii.) $\dfrac{1+\dfrac{a}{b}}{\dfrac{b}{a}-1} \div \dfrac{a^2-b^2}{1-\dfrac{2b}{a}+\dfrac{b^2}{a^2}}$

Find the value of the former expression when

$$x = -1 \text{ and } y = \frac{1}{4}$$

(5) Solve the following equations

(i.) $\dfrac{x}{2} - \dfrac{0{\cdot}05x-7{\cdot}5}{0{\cdot}6} = \dfrac{0{\cdot}25x+3{\cdot}8}{0{\cdot}3}$

(ii.) $\dfrac{x-y}{2} - \dfrac{3x-\frac{1}{2}y}{5} = \dfrac{1}{2}$

$$\frac{x}{3}-\frac{y}{2} = 2$$

(6) If $z^2=a-\dfrac{a^2}{y^2}$, and $y^2=a-\dfrac{a^2}{x^2}$, find the value of $a-\dfrac{a^2}{z^2}$ in terms of x.

(7) Find two numbers whose sum is 30 and whose difference equals a third part of the greater.

LIV.

(1) If a certain number be represented by x, write down expressions for

(i.) Its square.
(ii.) Its cube.

(iii.) The double of its square.
(iv.) The square of its double.
(v.) The sum of its square and its cube.
(vi.) The square of its cube.

Taking a straight line to represent the number x, draw diagrams to illustrate the difference between (iii.) and (iv.).

(2) Multiply $b^2-a^2+c^2+2bc$ by $2bc+a^2-c^2-b^2$

(3) Divide

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2 \text{ by } a^2+2ac+c^2-b^2$$

(4) Simplify

(i.) $(x+4)^3+3(x+4)(x+3)^2-(x+3)^3-3(x+4)^2(x+3)$

(ii.) $\dfrac{ax+x^2}{a^2-x^2}+\dfrac{a-x}{a+x}-\dfrac{a}{x}$

(iii.) $8a-2\{a-3[2a-4(5a-6\,.\,\overline{a-1})]\}$

(iv.) $\dfrac{x-y}{1+xy}+\dfrac{y-z}{1+yz}+\dfrac{z+x}{1+zx}$

(5) Find G.C.M. and L.C.M. of

$$x^4-3x^2+2 \text{ and } x^3-2x+1$$

(6) Solve the equations

(i.) $\dfrac{4x-5}{3}+2x=\dfrac{6x-2}{5}+3$

(ii.) $\dfrac{a-b}{x-c}=\dfrac{a+b}{x+2c}$

(iii.) $$\begin{aligned} x+y+z&=6\\ x+2y+3z&=14\\ x+4y+9z&=36 \end{aligned}$$

(7) If A does a piece of work in 10 days which A and B can do together in 7 days, how long would B take to do it alone?

LV.

(1) If $x=2$ and $y=-\frac{1}{2}$, find the numerical values of

$$3(x+y)-2(x-y),\ x^4y^6,\ x^3y^5, \text{ and } \frac{x^3-y^3}{x-y}$$

(2) Subtract $6x+1-2(x+5y)-\{2-(x+2y-1)\}$ from $2(2-y)-7x-2\{1-3y+6(y-x)\}$

(3) Divide

$$(a+b+c)^3-a^3-b^3-c^3 \text{ by } (a+b)(b+c)$$

(4) Simplify

$$2x^2-3\left(\frac{y^2}{2}-x^2\right)-\frac{1}{2}\left\{4x^2-\frac{1}{2}\left(24x^2+5y^2\right)\right\}$$

and divide the simplified expression by $x+\frac{y}{6}$

(5) Find the G.C.M. of

$$x^3+3x^2-25x+21 \text{ and } 2x^3-9x^2+10x-3$$

and write down these expressions in factors.

(6) Solve the equations

(i.) $(2x-1)(3x-2)-(x-1)(x+1)=5(x-1)^2$

(ii.) $$\begin{aligned} 0{\cdot}5x-0{\cdot}6y&=1{\cdot}38\\ \frac{x}{3}+\frac{y}{0{\cdot}5}&=2\end{aligned}$$

(7) Divide 279 into two parts such that one-third of the first part is less by 15 than one-fifth of the second part.

LVI.

(1) Multiply together

$$a-2b+1,\ 2a+b-1, \text{ and } 2a+3b+1$$

(2) Divide

$$8x^4-16x^3y-30x^2y^2+36xy^3 \text{ by } 4x^3-13xy^2-6y^3$$

(3) Verify your answer to the previous question by dividing in succession by $x-2y$, $2x+3y$, and $2x+y$ the factors of $4x^3-13xy^2-6y^3$; obtaining the same remainder as before.

(4) If $s=a+b+c$, show that

$$(s-a)^3+(s-b)^3+(s-c)^3+3abc$$

is divisible by $a+b+c$

(5) Simplify

$$\frac{\dfrac{a+b}{a+c}-\dfrac{a-b}{a-c}}{\dfrac{a-c}{a-b}-\dfrac{a+c}{a+b}}$$

(6) Solve the equations

(i.) $9(x+3)^2+16(x+1)^2=(5x+9)^2$

(ii.) $\dfrac{x-a}{x-a-c}+\dfrac{x-b}{x-b+c}=\dfrac{a}{a+c}+\dfrac{b}{b-c}$

(iii.) $\dfrac{3x+4y-8}{2}=\dfrac{5x+3y+1}{3}=12$

(7) If firstly the difference of two numbers be multiplied by the sum of their cubes, and secondly the sum of the same numbers be multiplied by the difference of their cubes; and then one result be subtracted from the other, the remainder will be a multiple of both the original numbers and also of their sum and difference.

LVII.

(1) Find the value of

$$x^4-5x^3+12x^2-13x+7 \text{ when } 2x+1=4$$

(2) Simplify

$$\left(x-\frac{y+z}{2}\right)\left(y+z-\frac{x}{2}\right)+\left(y-\frac{z+x}{2}\right)\left(z+x-\frac{y}{2}\right)$$
$$+\left(z-\frac{x+y}{2}\right)\left(x+y-\frac{z}{2}\right)$$

and show that the resulting expression is always negative.

(3) Factorize

$$(pb-qa)^2+(qc-rb)^2+(ra-pc)^2+(pa+qb+rc)^2$$

(4) Simplify

(i.) $\dfrac{8b+9x}{3b^2+13bx+12x^2}+\dfrac{7b+9x}{3b^2+7bx+4x^2}-\dfrac{4b}{b^2-x^2}$

(ii.) $\dfrac{4x(x-y+z)+(y-z)^2}{(x-y)y+(x+z)x+(x+z)^2}$

(5) Solve the equations

(i.) $\dfrac{3x-8}{25}+\dfrac{6x-3}{15}=\dfrac{8x-3}{60}+\dfrac{5x-7}{12}$

(ii.) $\left.\begin{array}{r}\dfrac{3x-5y}{8}+\dfrac{7y+4}{18}=\dfrac{8x-1}{15}-\dfrac{31}{24}\\ 16x-23y=11(x+y-3)\end{array}\right\}$

(6) Write down an expression of the fourth degree in x which leaves a remainder 1, whether it is divided by $x+1$, $2x+3$, or $3x+2$.

(7) A cyclist travels the first half of the distance from P to Q at the rate of 10 miles per hour; he then rests 15 minutes, and travels the remainder of the distance at the rate of 15 miles per hour. Had he travelled the whole distance without stopping at the rate of 11 miles per hour, he would have taken exactly the same time. Find the distance from P to Q.

LVIII.

(1) If $a=4$, $b=5$, $c=7$, $x=\frac{1}{2}$, $y=\frac{1}{3}$, find the numerical value of

$$\frac{c^2+2ab-a^2-b^2}{a^2-b^2+c^2+2ac} \text{ and of } \frac{ax+(a+b)y-(b+c)xy}{(b-c)x+(c-a)y}$$

(2) Multiply $x^3 - ax^2 - 2a^2x + a^3$ by $x^2 + ax - a^2$ and divide $1 - 5x^4 + 4x^5$ by $(1 - x)^2$

(3) Find the L.C.M. of $x^3 - 1$, $x^4 - 1$, and $x^6 - 1$

(4) Add together the fractions

$$\frac{1}{x - 2a}, \quad \frac{1}{x + 2a}, \quad \frac{8a^2}{4a^2x - x^3}$$

(5) Simplify

$$\frac{a^3 + b^3 + c^3 - 3abc}{(2a + 3b + c)(b - c) + (2b + 3c + a)(c - a) + (2c + 3a + b)(a - b)}$$

(6) Solve the equations

$$\text{(i.)} \quad \frac{x^2 - 5}{x - 1} + \frac{3x + 2}{4} = \frac{5x + 2}{6} + \frac{11x + 2}{12}$$

$$\text{(ii.)} \begin{cases} \dfrac{x - a}{2} + \dfrac{y - b}{3} = a \\[2ex] \dfrac{x - b}{3} + \dfrac{y - a}{2} = b \end{cases}$$

(7) A man divides a certain sum of money amongst his 3 children in shares proportional to their ages, the age of the eldest being three times that of the youngest. The next year he similarly divides an equal sum, the age of the second child being now double that of the youngest. The eldest now receives £1 13s. less, and the youngest £1 19s. more than before. What was the sum divided on each occasion?

LIX.

(1) Simplify the expression $(x - y)^3 - (x + y)^3 + 3xy$ and find its numerical value when $x = \frac{1}{2}$, $y = -\frac{1}{3}$

(2) Multiply together $x^2 - 3x + 3$, $x^2 + 3$, $x^2 + 3x + 3$

(3) Find the quotient and remainder when $x^4 - 3x^3 + 5x^2 - 4$ is divided by $x^2 - 4x + 3$

(4) Simplify

(i.) $x - 2[x - 3\{x - 4(x - \overline{5 - x})\}]$

(ii.) $\dfrac{1}{1-x} + \dfrac{1}{1+x} - \dfrac{2}{1+x^2} - \dfrac{4x^6}{1-x^4}$

(5) Resolve into factors

(i.) $64a^6 - 1$

(ii.) $p^2 + p(m-n) - mn$

(iii.) $(x-3)(x-4)(x-5) + (x-1)(x-5)(x-9)$

(iv.) $x^4 + 4$

(v.) $x^4 + 2x^3 - 4x - 4$

(6) Solve the equations

(i.) $(x-3)(x+4) = (x-2)(x+6)$

(ii.) $\dfrac{a+x}{b} + \dfrac{b-x}{a} = 2$

(iii.) $3x - 5y = 2x + 3y = 19$

(7) A bill of six guineas is paid in sovereigns, half-crowns, and shillings. There are 26 coins altogether, and there are four times as many half-crowns as sovereigns. Find the number of each coin.

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